Arbitrage Pricing

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The Basic Principle

Payoff equivalence and arbitrage opportunities

- Some features of transaction prices are predictable without explicitly solving the trading game.
- Perhaps the most important one comes from the concept of *arbitrage*, which in turn is based on the concept of *payoff equivalence*.
- Two bundles of securities are payoff equivalent if they have the same probability distribution that determines their payoffs at the end of the trading game. (When there are other forms of uncertainty of concern we require payoff equivalent securities to share the same conditional distribution.)
- Loosely speaking, an *arbitrage opportunity* exists if two payoff equivalent securities do not trade at the same price.
- In a limit order market there is an arbitrage opportunity if the ask price on one security is less than the bid price on the other, and there are investors who are indifferent between securities that are payoff equivalent.

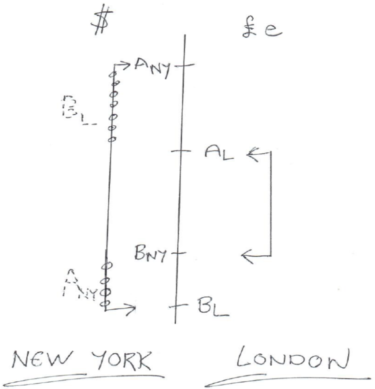
The Basic Principle

Equalizing prices across multiple markets

- Arbitrage opportunities attract traders whose profitable actions dissolve the opportunity.
- As they buy at the ask price and sell at the bid price, the ask price rises and the bid price falls to the point at which they cross and eliminate the arbitrage opportunity.
- Thus it should not be possible, by means of *market orders alone*, to sell one bundle of securities and purchase another payoff equivalent bundle for a net profit.
- Consequently the (highest) bid of one security should not exceed the (lowest) ask of a payoff equivalent security in a limit order market.
- This powerful principle imposes discipline on market orders in financial markets.

The Basic Principle
Illustrating the principle

Stock trading on
two exchanges



$e \equiv \frac{\$}{£}$: foreign exchange rate

Arbitrage Pricing in a Perfectly Liquid Market

A simple example

- Many of the ideas of arbitrage pricing can be conveyed by considering a simple example in a perfectly liquid market.
- Consider two securities that are defined on two independent coin tosses:
    1. One security pays one dollar if the coin comes up "heads" the first time.
    2. The other security pays one dollar if the same coin comes up "tails" the second time.
- We might imagine that:
    1. the two tosses stand for different periods
    2. "heads" means the firm had favorable news, and "tails" unfavorable.
- It is very easy to extend this example so that:
    1. the odds are not fifty/fifty
    2. there are more than two outcomes.

Arbitrage Pricing in a Perfectly Liquid Market

The example continues

- Note these securities do not have the same dividend stream.
- There are four possible outcomes:

Payoff matrix	HH	HT	TH	TT
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
First security	1	1	0	0
Second security	0	1	0	1

- Nevertheless they are payoff equivalent, because the distribution of returns is the same: in both cases there is 50 percent chance of receiving a dollar.

Payoff probability matrix by amount	0	1
First security	$\frac{1}{2}$	$\frac{1}{2}$
Second security	$\frac{1}{2}$	$\frac{1}{2}$

Arbitrage Pricing in a Perfectly Liquid Market

Applying the theory

- Arbitrage Pricing Theory (APT) predicts that in this situation (when the investors holds just one of the two assets) the prices of these two securities should be the same:

$$price\left(H \bigcup T, T\right) = price\left(H, H \bigcup T\right)$$

- Intuitively, the argument is that an arbitrage opportunity exists in a perfectly liquid market if the prices of the two assets differ:
  1. Traders holding the more expensive assets sell it, and buy a unit of the cheaper security.
  2. Their risk exposure is unchanged, but you pocket the price difference.
  3. These transactions cause the price of the more expensive security falls, and the price of the cheaper security rises.
  4. The arbitrage opportunity exists unless prices are equalized.
- APT therefore predicts that both securities trade at the same price *regardless of risk preference*.

Arbitrage Pricing in a Perfectly Liquid Market

Payoff equivalence applies to the full portfolio of securities, not to individual securities

- Now suppose you initially have one unit of each security.
- Would you have the same payoff distribution if you sold your unit of the first security and bought another unit of the second security?

Payoff matrix by state	HH	HT	TH	TT
Probability of state	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Before transaction	1	2	0	1
After transaction	0	2	0	2

Payoff probability matrix by amount	0	1	2
Probabilities before transaction	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
Probabilities after transaction	$\frac{1}{2}$	0	$\frac{1}{2}$

- 2 units of the second is not payoff equivalent to one unit of each.
- In this case the transaction exposes the trader to greater risk.

Rule: *Payoff equivalence (and risk exposure) applies to your total portfolio, not to a security considered in isolation.*

- *Hedges* are securities that eliminate (or more generally) reduce the dispersion of payoffs due to uncertainty.
- As before suppose you have one unit of each security.
- Consider the following portfolio adjustment:
  1. Buy a derivative that pays one dollar if the tosses go (tails, heads)
  2. Sell a derivative that pays one dollar if the tosses go (heads, tails).

Payoff matrix by state	HH	HT	TH	TT
Before transaction	1	2	0	1
Buy first derivative	0	0	1	0
Sell second derivative	0	-1	0	0
After hedge transaction	1	1	1	1

- These two trades are called a (perfect) hedge since they guarantee you will be paid exactly one dollar.

- Notice that the hedge does not require you to know the probabilities.
- However you do need to know all the payoff relevant states.
- Arbitrage pricing theory implies:

$$
\begin{aligned}
1 &= price\left(H \bigcup T, T\right) + price\left(H, H \bigcup T\right) \\
&\quad + price\,(T, H) - price\,(H, T)
\end{aligned}
$$

- In words, the price of the hedge is the difference between the price of the sure thing and the price of the security itself.

Hedging in a Perfectly Liquid Market

Using APT to price a third derivative

- To complete this example, suppose you only hold a third derivative, which pays one dollar if the tosses produce one head and one tail.
- What is it worth?
- Consider adjusting your portfolio as indicated by the following matrix:

Payoff matrix by state	*HH*	*HT*	*TH*	*TT*
Third derivative	0	1	1	0
Sell first derivative	0	0	-1	0
Sell second derivative	0	-1	0	0
After hedge transaction	0	0	0	0

- APT implies:

$$price\left[(H,T)\bigcup(T,H)\right] = price\,(T,H) + price\,(H,T)$$

Hedging in a Perfectly Liquid Market

Another way to price the third derivative using APT

- Another way of pricing the third security uses a different combination of the existing assets:

Payoff matrix by state	HH	HT	TH	TT
Third derivative	0	1	1	0
Sell sure thing	-1	-1	-1	-1
Buy second security	0	1	0	1
Buy first security	1	1	0	0
Sell two second derivatives	0	-2	0	0
After hedge transaction	0	0	0	0

- Thus APT implies:

$$\begin{aligned} & price\left[(H,T)\bigcup(T,H)\right] \\ = \;& 1 - price\left(H\bigcup T, T\right) - price\left(H, H\bigcup T\right) + 2price\left(H,T\right) \end{aligned}$$

- The powerful principle of arbitrage directly applies to market orders in the financial sector.
- But it also imposes a harsh discipline on commodity markets too.
- For example:
    - The prices of oil, coal and natural gas move together because they are such close substitutes in the production of energy.
    - When the price of extracting natural gas declines, electric utilities that can cheaply convert from coal to gas power, do so.
    - This leads directly to the closure of the coal mines that are most expensive to operate.
    - Thus arbitrage in commodity markets has a direct impact on income distribution, unemployment and wealth.

- Suppose there are J assets labeled as $j \in \{1, 2, \ldots, J\}$.
- Consider two payoff equivalent portfolios, denoted by $x \equiv (x_1, \ldots, x_J)$ and $y \equiv (y_1, \ldots, y_J)$.
- Suppose the trader owns x and wishes to adjust to y by making market orders only.
- If $y_j > x_j$ then the trader buys asset j and pays the ask price(s) A_j.
- If $y_j < x_j$ then the trader sells asset j and pays the bid price(s) B_j.
- Let $p_j > 0$ denote the price(s) the trader pays on the j^{th} asset where:

$$p_j \equiv \begin{cases} A_j \text{ if } y_j > x_j \\ B_j \text{ if } y_j < x_j \end{cases}$$

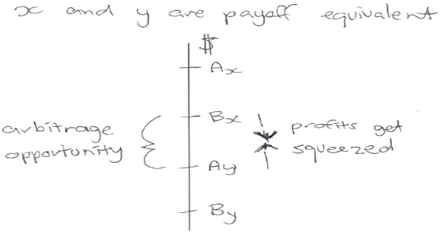
- Suppose it is profitable to sell x with market orders and buy y with markets orders:

$$\sum_{j=1}^{J} p_j \left(y_j - x_j\right) < 0$$

- A_j increases when traders buy y_j, while B_j falls as they sell x_j.
- This squeezes their profits from both ends.
- The *Arbitrage Pricing Theory* (APT) says arbitrage opportunities do not exist in equilibrium.
- That is, for any two payoff equivalent portfolios, denoted by x and y:

$$\sum_{j=1}^{J} p_j \left(y_j - x_j\right) \geq 0$$

- In this illustration the trader sells x at B_x (the bid) and buys y at A_y (the ask), for a marginal net gain of $B_x - A_y$.
- This causes B_x to fall and A_y to rise, shrinking the gain:



- Note selling y at B_y and buying x at A_x, yields losses of $A_x - B_y$.
- More generally:

$$\sum_{j=1}^{J} p_j \left(y_j - x_j\right) < 0 \Longrightarrow \sum_{j=1}^{J} p_j \left(x_j - y_j\right) > 0$$

- The previous slide showed that if $x \equiv (x_1, \ldots, x_J)$ and $y \equiv (y_1, \ldots, y_J)$ are payoff equivalent APT implies:

$$\sum_{j=1}^{J} p_j \left(y_j - x_j \right) \geq 0$$

- If all the markets are perfectly liquid, then $A_j = B_j$. In that case, selling y and to obtain x is the negative of selling x and to obtain y.
- Since APT implies selling y and to obtain x cannot be profitable:

$$\sum_{j=1}^{J} p_j \left(x_j - y_j \right) \geq 0$$

- Putting the two inequalities together we obtain:

$$\sum_{j=1}^{J} p_j \left(y_j - x_j \right) = 0$$

Efficient Markets Hypothesis

A general formulation

- For an expected value maximizer, who by definition only occurs about the mean of a payoff distribution, the *Efficient Markets Hypothesis* (EMH) is essentially a corollary of APT.
- Suppose the j^{th} asset only pays out upon liquidation and let A_{jt} denote its ask price at time t, and B_{jt} its bid price.
- We also write $E\left[\cdot\,|Info_t\right]$ for the expected value of a random variable when know $Info_t$ about it. The EMH says:

$$E\left[B_{j,t+1}\,|Info_t\right] \leq A_{jt}$$

$$B_{jt} \leq E\left[A_{j,t+1}\,|Info_t\right]$$

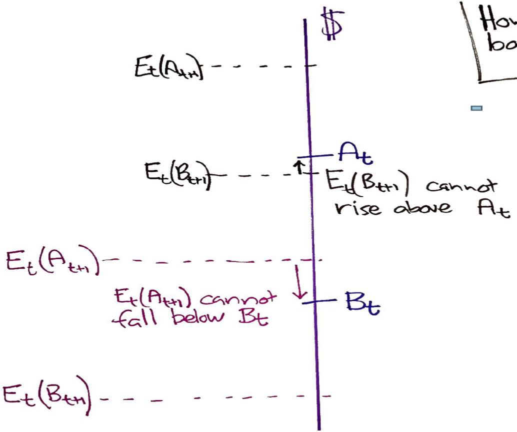
- Using the law of iterated expectations it is easy to generalize these inequalities to:

$$E\left[B_{j,t+s}\,|Info_t\right] \leq A_{jt}$$

$$B_{jt} \leq E\left[A_{j,t+s}\,|Info_t\right]$$

Efficient Markets Hypothesis

Illustrating the bounds of the expected spread



How much does EMH bound the expected spread?

Efficient Markets Hypothesis

Perfectly liquid markets

- In a perfectly liquid market, the spread collapses and $A_{jt} = B_{jt}$.
- From the previous slide, EMH implies:

$$E\left[B_{j,t+1}\,|Info_t\right] \leq A_{jt} = B_{jt} \leq E\left[A_{j,t+1}\,|Info_t\right]$$

- But if $A_{j,t+1} = B_{j,t+1}$ then:

$$E\left[B_{j,t+1}\,|Info_t\right] = E\left[A_{j,t+1}\,|Info_t\right]$$

- So the weak inequalities above can only be satisfied if:

$$E\left[B_{j,t+1}\,|Info_t\right] = A_{jt} = B_{jt} = E\left[A_{j,t+1}\,|Info_t\right]$$

.

- In words, transaction prices follow a random walk.
- The hypothesis that asset prices follow a random walk might be regarded as a test of perfect liquidity.

Simplifying the framework for analysis

A recapitulation

- Arbitrage pricing theory (APT) can be stated limit order markets
- Inequalities rather than equalities define the equilibrium linkages between payoff equivalent portfolios.
- More generally, arbitrage can only occur in one direction, and its marginal benefit declines with scale.
- Similarly the Efficient Markets Hypothesis (EMH) can be restated within a limit order market, again in the weaker form of inequalities rather than equalities.
- When markets are perfectly liquid, EMH predicts that prices follow a random walk when traders are risk neutral.

Simplifying the framework for analysis

The building blocks of modern portfolio investment

- APT provides some guidance about how assets closely related to each other are priced.
- But how do buyers and sellers form their individual valuations that ultimately determine the terms of trade?
- The notion of competitive equilibrium is the economist's standard tool for reconciling demand and supply in almost every market imaginable.
- Therefore it should come as no surprise that portfolio investment strategy is based on models of competitive equilibrium.
- Modern portfolio theory is based on three fundamental components:
    1. competitive equilibrium (or perfect liquidity).
    2. investor attitudes towards uncertainty and risk.
    3. the rate at which investors are willing to sacrifice current consumption for the certain gain of future consumption.