Portfolio Investment

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Trade and Investment Strategy

December 2023

Introduction

Simplifying the framework for analysis

- *Competitive equilibrium* is commonly assumed in portfolio analysis, widely used to finesse descriptions of institutional detail:
    1. Markets are *perfectly liquid*. (Bid equals ask)
    2. Individual order size does not affect prices. (*Price taking* behavior)
    3. There are neither unfilled orders nor excess inventories. (*Markets clear*)
- Note the definition does not answer:
    1. How is the competitive equilibrium price formed?
    2. Why does every trader believe he or she cannot influence price?
    3. Does a competitive equilibrium price exist?

- The first two conditions of competitive equilibrium imply that investors are price takers, and are most plausible for small investors.
- Large trades typically affect the spread, and therefore **block traders** should also account for the effects of their trading activity on prices they pay.
- About half the trading in financial securities of publicly listed firms are large trades (over $200,000), many executing off the limit order books.
- We now derive the fundamental equation that characterizes:
  - the trade-off at the individual level between current and future consumption in terms of asset returns
  - a **personalized mean variance frontier** for small investors.

- Suppose there are J financial securities, and let q_{tj} denote the amount of the j^{th} security owned at the end of period t.
- Let π_{tj} denote the physical firm growth underlying the j^{th} security revealed at the beginning of period t and applied to the securities carried from the previous period.
- That is. holding q_{tj} unit of j at the end of t produces $\pi_{t+1,j}q_{tj}$ when period $t+1$ begins.
- Assume markets are perfectly liquid, and let p_{tj} denote the price of (buying or selling) the j^{th} security in period t, measured in consumption units.
- For example, imagine you can harvest or prune vegetation for heat and food, or let it grow more, planting its seed and fruit.
- Note that $p_{tj} \geq 1$ since individuals can consume (liquidate) the asset.

A Model of Portfolio Choice

Law of motion for wealth

- Individuals also receive wages at the beginning of each period t.
- Let w_t denote real wages (measured in consumption units).
- Also let m_t denote real money balances (liquid non-interest bearing assets) held over from one period to the next.
- Upon entering the current period, the individual receives a return on assets she holds, receives a wage, trades, consumes, and carries some assets and cash over to the next period.
- The *law of motion for her wealth* is:

$$c_t + m_t + \sum_{j=1}^{J} p_{tj} q_{tj} \leq w_t + m_{t-1} + \sum_{j=1}^{J} p_{tj} \pi_{tj} q_{t-1,j}$$

- The individual makes choices in periods $t \in \{1, \ldots, T\}$.
- Let $E_t[\cdot]$ denote the expectation based on information period t.
- At each time $t < T$ she faces a sequence of constraints in future periods $s \in \{t, \ldots, T-1\}$:

$$c_s \leq w_s + (m_{s-1} - m_t) + \sum_{j=1}^{J} p_{sj} \left(\pi_{sj} q_{s-1,j} - q_{sj} \right) \tag{1}$$

- Given her current wealth $w_t + \sum_{j=1}^{J} p_{tj} \pi_{tj} q_{t-1,j}$ she chooses $(c_t, m_t, q_{t1}, ..., q_{tJ})$ to maximize:

$$u(c_t) + E_t \left[\sum_{s=t+1}^{T} \beta^{s-t} u(c_s) \right]$$

subject to all the constraints indicated by (1).

A Model of Portfolio Choice

Portfolio choices

- Here, bequeathing heirs counts as consumption.
- Since $u(c_t)$ is strictly increasing, all wealth is consumed, so:

$$c_T = w_T + m_{T-1} + \sum_{j=1}^{J} p_{Tj} \pi_{Tj} q_{T-1,j}$$

and for all $t < T$:

$$c_t = w_t + (m_{t-1} - m_t) + \sum_{j=1}^{J} p_{tj} \left(\pi_{tj} q_{t-1,j} - q_{tj} \right)$$

- Substituting for consumption sequence $(c_t, c_{t+1}, \ldots, c_T)$ in the utility functions, expected lifetime utility from period t onwards is:

$$u\left(w_t + m_{t-1} - m_t + \sum_{j=1}^{J} p_{tj} \left(\pi_{tj} q_{t-1,j} - q_{tj} \right) \right) \tag{2}$$
$$+ E_t \left[\sum_{s=t+1}^{T} \beta^{s-t} u \left(\begin{array}{l} w_s + (m_{s-1} - m_s) \\ + \sum_{j=1}^{J} p_{sj} \left(\pi_{sj} q_{s-1,j} - q_{sj} \right) \end{array} \right) \right]$$

- This only leaves the asset choices $(m_t, q_{t1}, ..., q_{tJ})$ to solve for.
- Notice that q_{tk} only enters (2) in two spots, namely:

$$u\left(w_t + m_{t-1} - m_t + \sum_{j=1}^{J} p_{tj}\left(\pi_{tj} q_{t-1,j} - q_{tj}\right)\right)$$
$$\text{and } u\left(w_t + m_{t-1} - m_t + \sum_{j=1}^{J} p_{tj}\left(\pi_{tj} q_{t-1,j} - q_{tj}\right)\right)$$

- The *interior first order condition* (FOC) for $k \in \{1, \ldots, J\}$ is:

$$p_{tk} u'\left(w_t + m_{t-1} - m_t + \sum_{j=1}^{J} p_{tj}\left(\pi_{tj} q_{t-1,j} - q_{tj}\right)\right) \tag{3}$$
$$= E_t\left[p_{t+1,k}\pi_{t+1,k}\beta u'\left(\begin{array}{c} w_{t+1} + (m_t - m_{t+1}) \\ + \sum_{j=1}^{J} p_{t+1,j}\left(\pi_{t+1,j} q_{tj} - q_{t+1,j}\right) \end{array}\right)\right]$$

Optimal Choices

How many units of the asset do you hold?

- Define the real rate of return on asset k at the beginning of t by:

$$r_{tk} \equiv \frac{p_{tk}\pi_{tk}}{p_{t-1,k}}$$

- Noting again that:

$$c_t = w_t + (m_{t-1} - m_t) + \sum_{j=1}^{J} p_{tj}\left(\pi_{tj} q_{t-1,j} - q_{tj}\right)$$

the FOC, that is (3), requires:

$$u'(c_t) = E_t\left[r_{t+1,k}\beta u'(c_{t+1})\right]$$

- Note how similar (3) is to the FOCs for analyzing:
  - risk and uncertainty from holding different assets (Lecture 11).
  - current and future consumption (Lecture 10).

Optimal Choices

When should you liquidate the asset?

- Since $u(c)$ is a concave increasing function it follows that if no units of the k^{th} asset are held, that is $q_{tk} = 0$ then:

$$u'(c_t) > E_t\left[r_{t+1,k}\beta u'(c_{t+1})\right]$$

The marginal utility of current consumption $u'(c_t)$ is too high to justify acquiring k given the expected benefits of its return on consumption next period.

- Define W_t, disposable wealth (including wage income) at the beginning of the period t as:

$$W_t \equiv w_t + m_{t-1} + \sum_{j=1}^{J} p_{tj} \pi_{tj} q_{t-1,j}$$

- Then (3) can be expressed as:

$$\begin{aligned} & u'\left(W_t - m_t - \sum_{j=1}^{J} p_{tj} q_{tj}\right) \\ = \; & E_t\left[r_{t+1,k} \beta u'\left(W_{t+1} - m_{t+1} - \sum_{j=1}^{J} p_{t+1,j} q_{t+1,j}\right)\right] \end{aligned}$$

- Given W_t this optimality condition does not depend on the performance of individual assets in period t.

  For example if $r_{tk} > r_{tk'}$ but the probability distributions of $\pi_{t+1,k}$ and $\pi_{t+1,k'}$ do not change, then the individual typically sells units of k and buy units of k'.

- This is called **rebalancing**.

- Rearranging the FOC yields:

$$1 = E_t \left[\left(\frac{p_{t+1,k}\pi_{t+1,k}}{p_{tk}} \right) \beta \frac{u'(c_{t+1})}{u'(c_t)} \right] \equiv E_t \left[r_{t+1,k} MRS_{t+1} \right]$$

where:

$$MRS_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

- This is called the **fundamental equation of portfolio choice**.

The return on an asset return is discounted by the marginal rate of substitution between current and future consumption in expectation.

- When $u(c_t) = c_t^{1-\alpha}/(1-\alpha)$ the equation yields:

$$\beta E_t \left[r_{t+1,k} \left(\frac{c_t}{c_{t+1}} \right)^{\alpha} \right] = 1$$

- For example, in a world where there is no uncertainty, and r_{t+1} is the risk free rate, this equation simplifies to:

$$\beta r_{t+1} \left(\frac{c_t}{c_{t+1}} \right)^{\alpha} = 1$$

- And specializing to $u(c_t) = \ln(c_t)$ we obtain:

$$\beta E_t \left[r_{t+1,k} \left(\frac{c_t}{c_{t+1}} \right) \right] = 1$$

- Finally if $u(c_t) = c_t$ the expected return is equated with the interest rate for all interior choices:

$$\beta E_t \left[r_{t+1,k} \right] = 1$$

- Recall from the definition of a covariance:

$$\begin{aligned} cov\left(r_{t+1,k}, MRS_{t+1}\right) &= E_t\left[r_{t+1,k} MRS_{t+1}\right] - E_t\left[r_{t+1,k}\right] E_t\left[MRS_{t+1}\right] \\ &= 1 - E_t\left[r_{t+1,k}\right] E_t\left[MRS_{t+1}\right] \\ &= 1 - E_t\left[r_{t+1,k}\right] / r_{t+1} \end{aligned}$$

where:

  - the second line uses the fundamental equation of portfolio choice
  - the third line uses the definition of the risk free rate.

- Rearranging this equation gives the risk correction for the k^{th} asset:

$$E_t\left[r_{t+1,k}\right] - r_{t+1} = -r_{t+1} cov\left(r_{t+1,k}, MRS_{t+1}\right)$$

where $E_t\left[r_{t+1,k}\right] - r_{t+1}$ is the **expected excess return**.

- Dividing through by r_{t+1}:

$$\frac{E_t\left[r_{t+1,k}\right] - r_{t+1}}{r_{t+1}} = -cov\left(r_{t+1,k}, MRS_{t+1}\right)$$

The expected excess return on an asset as a proportion of the risk free rate equals the negative of the covariance between the asset return and the marginal rate of substitution between current and future consumption.

- When c_{t+1} and $r_{t+1,k}$ tend to move in opposite directions (like insurance payouts), then $cov\left(r_{t+1,k}, MRS_{t+1}\right)$ is positive and the investor accepts a mean return less than the risk free rate.
- When they mainly move together (like the market portfolio and wages in many occupations), then $cov\left(r_{t+1,k}, MRS_{t+1}\right)$ is negative and he requires more.

- The third defining characteristic of competitive equilibrium, market clearing, closes the model.
- Let $\{1, \ldots, N\}$ denote the population of individual agents, and write $r_{t,j}q_{t-1,j}^{(n)}$ for the amount of the j^{th} asset owned by agent n in period t:
  1. If $p_{tj} > 1$, then the j^{th} asset is not consumed by anybody in period t so market clearing means:
$$\sum_{n=1}^{N}\left(r_{t,j}q_{t-1,j}^{(n)} - q_{tj}^{(n)}\right) = 0$$
  2. If the j^{th} asset is not bought by anybody in period t, then it is consumed (liquidated) and $p_{tj} = 1$.
  3. All other assets, collectively denoted by A, are priced at one and are perfect substitutes in consumption. Market clearing only requires:
$$\sum_{n=1}^{N}\left(c_t^{(n)} - w_t^{(n)} + m_t^{(n)} - m_{t-1}^{(n)}\right) = \sum_{n=1}^{N}\sum_{j\in A}\left(r_{t,j}q_{t-1,j}^{(n)} - q_{tj}^{(n)}\right)$$

- If trading was a costless exercise, then investing through a mutual fund would be free.
- Consider investing through a personally tailored mutual fund which comprises the n^{th} investor's portfolio:

$$s_t^{(n)} = \sum_{j=1}^{J} p_{tj} q_{tj}^{(n)}$$

- Its real return is defined by:

$$\pi_{t+1,s} \equiv \left(\sum_{j=1}^{J} p_{t+1,j} q_{tj}^{(n)} \pi_{t+1,j} \right) / s_t^{(n)}$$

- The individual could economize on his transactions by purchasing shares in a mutual fund that replicate the financial portfolio he would have purchased himself.

Mutual funds also satisfy the fundamental equation of portfolio choice

- Weighting the first order condition by $q_{tj}^{(n)}$ and then summing over j gives:

$$u'\left(c_t\right) s_t^{(n)} = \sum_{j=1}^{J} p_{tj} q_{tj}^{(n)} u'\left(c_t\right) = \sum_{j=1}^{J} E_t\left[p_{t+1,j} q_{tj}^{(n)} \pi_{t+1,j} \beta u'\left(c_{t+1}\right)\right]$$

- Dividing both sides of this equation by $u'\left(c_t\right) s_t^{(n)}$ yields the fundamental equation of portfolio choice for the mutual fund:

$$1 = E_t\left[\sum_{j=1}^{J}\left(\frac{p_{t+1,j} q_{tj}^{(n)} \pi_{t+1,j}}{s_t^{(n)}}\right) \beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\right] \equiv E_t\left[r_{t+1,s} MRS_{t+1}\right]$$

- There are no tax implications. Rather than buying a whole portfolio of stocks, the investor simply picks the mutual with a return that most suits his marginal rate of substitution at the same time he decides how much to consume in the current period.

Financial Intermediation

Why don't individuals form their own portfolios?

- Since transaction fees per share decline with volume, and information about the risk characteristics of each stock is costly, investors with like minded strategies have an incentive to band together. Mutual funds also give investors a say on the board.
- Recall that with the same wealth and wage process (plus other demographics) investors with same utility function should hold the same portfolio. Thus mutual funds compete with each other by constructing portfolios that appeal to different niches in the market.
- This explains why the S & P composite of 1500 firms covers about 85 percent of the U.S. equity market, there are more than 8000 mutual funds to invest in.
- One cautionary remark is that investing through an intermediary does carry an additional risk of **fraud** and **incompetence**.

Financial Intermediation

Hedges

- Many companies routinely hedge on input prices, and insure themselves against financial events they cannot control.
- For example fuel is very large operating cost for airlines:
  - Southwest hedges against unanticipated changes in fuel prices.
  - American, United and Delta do not hedge.
  - Many long distance carriers specializing in long distance travel (British Airways, Singapore Airlines, Qantas) buy oil futures.
- Companies cannot hedge risks or insure themselves against negative financial events they influence, because this would reduce their incentives to avoid the risk, creating a moral hazard problem.
- But why should companies hedge any risks, since individual investors and the financial retail sector can replicate every hedging strategy for themselves?

Financial Intermediation

Why do companies hedge?

- When Southwest buys oil on the futures market, are they simply smoothing their cost revenue stream and returns stream?
- In Strategic Corporate Management (45-870) I argue that companies create value by concentrating on those activities which are more efficiently achieved inside the firm and outsource those activities that are easily verified where it lacks a comparative advantage.
- For example Southwest manages the risks associated with those activities about which it has specialized knowledge, such as route and seasonal demand, relations with its unionized workforce, and aircraft maintenance.
- If there are transactions costs and information imperfections, hedging and insurance reduces distractions, helps solidify the company's mission and branding strategy, makes the company's performance more dependent on its core business, thus increasing its transparency and value as a financial security.