Session 3.1
The Revenue Equivalence Theorem

This session provides a formal statement and intuitive proof of the reneu equivalence theorem. Part of the value from following the proof is to see what kinds of assumptions are used in obtaining this result.

Revenue equivalence theorem

- The revenue equivalence theorem states that in private value auctions, the expected surplus to each bidder does not depend on the auction mechanism itself providing the following conditions are satisfied:

1. Every bidder is risk-neutral.
2. Valuations are private and identically distributed.
3. In equilibrium the bidder with highest valuation wins.
4. The lowest possible valuation has zero expected value.

- Note that if all bidders obtain the same expected surplus, the auctioneer obtains the same expected revenue too.

Preferences and Expected Payoffs

- Let $Pr(v_n)$ denote the probability the n^{th} bidder with valuation v_n will win the auction when all players bid according to their equilibrium strategy.

- Let $C(v_n)$ denote the expected costs (including any fees to enter the auction, and payments in the case of submitting a winning bid).

- Let:

$$U(v_n) = Pr(v_n)\, v_n - C(v_n)$$

denote the expected net value of the n^{th} bidder from following his equilibrium strategy when everyone else does too.

A revealed preference argument

- Suppose the valuation of n is v_n and the valuation of j is v_j.
- The surplus from n bidding as if his valuation is v_j is $U(v_j)$, the value from participating if his valuation is v_j, plus the difference in how he values the expected winnings compared to a bidder with valuation v_j, or $(v_n - v_j)Pr(v_j)$.
- The value of n following his solution strategy is at least as profitable as deviating from it by pretending his valuation is v_j. Therefore:

$$U(v_n) \geqslant U(v_j) + (v_n - v_j)Pr(v_j)$$

Revealed preference continued

- For convenience, we rewrite the last slide on the previous page as:

$$U(v_n) - U(v_j) \geqslant (v_n - v_j)Pr(v_j)$$

- Now viewing the problem from the j^{th} bidder's perspective we see that by symmetry:

$$U(v_j) \geqslant U(v_n) + (v_j - v_n)Pr(v_n)$$

which can be expressed as:

$$(v_n - v_j)Pr(v_n) \geqslant U(v_n) - U(v_j)$$

A fundamental equality

- Putting the two inequalities together, we obtain:

$$(v_n - v_j)\,Pr(v_n) \geqslant U(v_n) - U(v_j) \geqslant (v_n - v_j)\,Pr(v_j)$$

- Writing:

$$v_n = v_j + dv$$

yields:

$$dU(v) = P(v)dv$$

which, upon integration, yields:

$$U(v_n) = U(\underline{v}) + \int_{v=\underline{v}}^{v_n} P(v)dv$$

- This last formula shows the surplus from winning the auction does not depend on the bidding rules, thus proving the revenue equivalence theorem.