Technical Appendix for Lecture 3

Moral Hazard

Moral hazard arises when the unobserved choices of one player affects the payoff he receives from a person he is contracting with. Since the player's choice is not observed by other parties to the contract, the contract cannot directly specify which choice should be taken. Linking the player's payments to the consequences of his action, can help align his incentives with those of the other players, even though the consequences are only partly attributable to or caused by the action itself. For example, managers are paid to make decisions on behalf of the shareholder interests they represent. If they were paid a flat rate, why would they pursue the objectives of shareholders? Lawyers representing clients are more likely to win if they are paid according to their record, and also whether they win the case in question or not. The extent of warranties against product defects may affect how a product is used, and how much care is taken.

Sometimes the unobserved action can be inferred exactly at some later point in time. If an air-conditioning unit is installed during winter, the guarantee should extend to the summer, so that the owner is compensated if the unit malfunctions during peak usage. Similarly car mechanics can be paid a fixed wage if there are also penalty provisions for poor workmanship that might only be revealed after the vehicle has been serviced. In these cases a moral hazard problem does not exist, providing the contract period covers sufficiently long warranty period.

A Model

Shareholders begin the game we now consider by making an offer to the manager. The manager can reject the offer by choosing action a_0 to be employed elsewhere (or not at all). If the manager accepts the offer he chooses one of two actions, neither of which are observed by the shareholders. Action a_1, called shirking, gives him more personal satisfaction. This action is not limited to the trade Action a_2, called working diligently, is the one the board of directors prefer the manager to take. This conflict between the board (or shareholders) is not meant to highlight the manager's trade-off between work and say, golf. It simply reflects the fact that their respective priorities are not automatically aligned. We denote by $u_j(w_j)$ the utility the manager receives from taking action a_j and receiving a wage of w_j for $j \in \{0,1,2\}$, and for convenience set $u_0(w_0) \equiv u_0$. We assume that $u_j(w)$ is concave increasing in w for each $j \in \{0,1,2\}$.

For example suppose that the manager's utility takes exponential form

$$u_j(w) \equiv -\alpha_j \exp(-\rho w)$$

where the positive constant ρ is called the coefficient of absolute risk aversion, and α_j is a positive constant which capture the manager's preference for choosing action a_j for $j \in \{0,1,2\}$. Noting utility is negative, we interpret a_j as a utility loss from choosing activity j. To assert the manager prefers shirking to working diligently is then equivalent to saying $\alpha_2 > \alpha_1$.

Monitoring the Manager

To help motivate the analysis we first assume that the actions of the manager can be directly monitored by shareholders. If risk neutral shareholders directly observe how hard the manager works, then the optimal contract with him is a fixed wage, since the manager is risk averse. To induce the manager to accept employment with the firm the participation constraint requires

$$u_0(w_0) \leq \max\{u_1(w_1), u_2(w_2)\}$$

and to induce him to work hard the shareholders must set w_1 and w_2 so that $u_1(w_1) < u_2(w_2)$. For example if the manager has an exponential utility function, in a world of full information incentive compatibility requires

$$u_1(w) - u_2(w) = (\alpha_2 - \alpha_1)\exp(-\rho w) = \exp\{-\rho[w + \rho^{-1}\log(\alpha_2 - \alpha_1)]\} \leq 0$$

The second equality shows the manager is willing to pay up to $\rho^{-1}\log(\alpha_2 - \alpha_1)$ to choose shirking over diligence if both activities are compensated the same way. In other words $\rho^{-1}\log(\alpha_2 - \alpha_1)$ is the premium shareholders must pay the manager to work hard rather than shirk. In this case the participation constraint requires

$$\alpha_2 \exp(-\rho w_2) \leq \alpha_0 \exp(-\rho w_0)$$

Taking logarithms and making w_2 the subject of the inequality we obtain

$$w_2 - w_0 \geq \rho^{-1}\log\left(\frac{\alpha_2}{\alpha_0}\right)$$

Thus $\rho^{-1}(\log\alpha_2 - \log\alpha_0)$ is compensating differential shareholders must pay the manager to work diligently for the firm that than accept employment elsewhere. If $\alpha_2 > \alpha_0$, then w_2 must exceed w_0 for the manager to take the job and work hard. Our discussion implies that in a world of full information, the minimum cost of employing the manger to work diligently is

$$w_2 = w_0 + \rho^{-1}\log\left(\frac{\alpha_2}{\alpha_0}\right)$$

and this can be achieved by setting a penalty of at least $\rho^{-1}(\log\alpha_2 - \log\alpha_1)$ for shirking. In this world there is no rationale for paying the manager on the basis of the firm's performance because the manager's performance can be fully monitored.

Signals

Instead of assuming the manager's activities are monitored, we now suppose that shareholders only observe a signal which is generated by a probability distribution that depends on the manager's choice. Let the random variable x denote the signal, and suppose $f_j(x)$ is the probability density function for x when the manager chooses action a_j for $j \in \{1,2\}$. The ratio of the two probability density functions, denoted $g(x)$, plays an important role in our analysis of this problem:

$$g(x) \equiv \frac{f_1(x)}{f_2(x)}$$

We interpret a realization of $g(x)$ as the likelihood that the manager shirked rather than worked hard. When shareholders receive a signal x^* they are inclined to believe the manager had shirked if $g(x^*) > 1,$ and conversely worked diligently if $g(x^*) < 1.$ We remark that $g(x)$ ranges from 0 to ∞ : if $g(x^*) = 0,$ shareholders conclude the manager worked diligently, whereas if $g(x^*) = \infty$ he surely shirked. One example of a signal is the firm's abnormal return. A reasonable goal for shareholders is to maximize the expected value of abnormal returns net of expected managerial compensation. In this case we assume that

$$E_2[x] \equiv \int_{-\infty}^{\infty} xf_2(x)dx > \int_{-\infty}^{\infty} xf_1(x)dx \equiv \int_{-\infty}^{\infty} g(x)f_2(x)dx \equiv E_2[xg(x)]$$

To further specialize, suppose that if the manager works diligently, abnormal returns are uniformly distributed on the closed interval between -1 and 1, which means $f_2(x) = 1/2$ for $x \in [-1,1],$ but that if he shirks the cumulative distribution function has a triangular shape on the same support, $f_1(x)$ taking the from $(1-x)/2$ for $x \in [-1,1].$ Then $g(x) = f_1(x),$ so the likelihood ratio is monotonically declining in x with $g(x) > 1$ if and only if $x < 0.$ In this specialization $E_2[x] = 0$ but $E_1[x] = -1/3.$

When $g(x^*)$ is finite shareholders cannot deduce from the realization x^* whether the manager worked diligently or not. Nevertheless there exist pairs of probability density functions $f_1(x)$ and $f_2(x)$ that allow shareholders to implement the outcome implied by the full information contract derived above even though they only observe the signal, not the manager's action. If there is a strictly positive probability that the manager will be caught shirking, meaning $g(x^*) = \infty$ for some values of x^* that cannot be reached if the manager works diligently, then threatening him with a very high penalty will deter him from shirking. A two part contract is optimal, comprising a constant wage w that meets the participation constraint, supplemented by a penalty that is incurred if the manger is caught shirking. This contract fully insures the manager, and achieves the same first best solution that could be attained if the action is observed and contracted upon.

For example suppose that $f_2(x)$ is defined as before, uniform on $[-1,1],$ but that $f_1(x) = 1/3$ with support $x \in [-2,1].$ Then $g(x) = 2/3$ on the interval $[-1,1]$ but is unbounded on the interval $[-2,-1).$ If the manager shirks, he will be caught one third of the time. By posting a sufficiently high penalty for sufficiently low signals $x < -1,$ he is deterred from shirking even if his wage is constant on the interval $x \in [-1,1].$ This contract form ensures the manager and the shareholders are as well off as they would be if the shareholders monitored the manager's performance. Although optimal compensation depends on the realization of the signal, a random variable, in this particular example the manager receives a certain wage conditional on diligent work, even though diligence cannot be verified in every state.

Constraints on Acceptable Contracts

A moral hazard problem arises only if the support of $f_1(x)$ is contained in the support of $f_2(x)$. When $g(x)$ is a finite valued function the optimal contract entails the manager accepting some risk. The compensation he receives might depend on the signal, and to indicate that dependence we now write $w_j(x)$ for the manager's compensation when he chooses action $j \in \{1,2\}$ and the shareholders subsequently observe the signal x. As above, there are two restrictions on contracts the manager accepts, the incentive compatibility and participation constraints. The participation constraint is now expressed as

$$E_j[u_j(w_j(x))] \geq u_0$$

for $j \in \{1,2\}$. In the exponential utility example the participation constraint is satisfied if and only if:

$$E_j\{\exp[\rho(w_j(x) - w_0)]\} \geq \left(\frac{\alpha_j}{\alpha_0}\right)$$

The incentive compatibility inducing diligent work is

$$E_2[u_2(w_2(x))] \geq E_1[u_1(w_1(x))]$$

or using the definition of $g(x)$:

$$E_2[u_2(w(x)) - u_1(w(x))g(x)] \geq 0$$

Given exponential utility this condition is

$$E_2\{\alpha_1 \exp[-\rho(w(x) - w_0)]g(x) - \alpha_2 \exp[-\rho(w(x) - w_0)]\} \geq 0$$

where we have multiplied both sides of the equation by the constant $\exp[\rho w_0]$ to express the condition in terms of deviations of $w(x)$ from w_0. This inequality simplifies to

$$E_2\left\{\exp[-\rho(w(x) - w_0)]\left[g(x) - \left(\frac{\alpha_2}{\alpha_1}\right)\right]\right\} \geq 0$$

These two constraints must be satisfied for the manager to accept employment with the firm and work diligently.

Minimizing the Expected Cost of Compensating the Manager

Having characterized the two constraints that shareholders face we now derive the minimum expected cost to shareholders from employing the manager at either effort level. There is no incentive compatibility constraint for shirking, because that is what the manager prefers. Therefore the signal is ignored in this case, the manager is fully insured, and as in the full information case the cost of achieving low effort is found by setting $u_1(w_1) = u_0$. In the exponential case, the compensation for shirking is set at

$$w_1 = w_0 - \rho^{-1}\log(\alpha_1/\alpha_0)$$

The minimum cost contract for engaging the manager to work diligently is found by choosing w for each x to minimize the expected cost of managerial compensation subject to the participation and incentive compatibility constraints. The Lagrangian for this problem can be expressed as

$$E\{w(x) - \lambda_1[u_0 - u_2(w(x))] - \lambda_2[u_2(w(x)) - g(x)u_1(w(x))]\}$$

where λ_1 is the Lagrange multiplier for the participation constraint, and λ_2 is the Lagrange multiplier for the incentive compatibility constraint. In principle the first order condition can be solved for $w(x)$ along with the Lagrange multipliers λ_1 and λ_2 using the participation and incentive compatibility constraints. In the formula above we have dropped the subscript 2 on both the expectation operator $E_2[\bullet]$ and the wage function $w_2(x)$ to simplify the notation a little.

For example when the manager has exponential utility, the Lagrangian becomes

$$E\left\{w(x) - \lambda_1\left[\exp[-\rho(w(x) - w_0)] - \left(\frac{\alpha_0}{\alpha_2}\right)\right] - \lambda_2 \exp[-\rho(w(x) - w_0)]\left[\left(\frac{\alpha_2}{\alpha_1}\right) - g(x)\right]\right\}$$

Differentiating with respect to $w(x)$ for each value of x the first order condition for this problem is:

$$1 + \rho\lambda_1 \exp[-\rho(w(x) - w_0)] + \rho\lambda_2 \exp[-\rho(w(x) - w_0)]\left[\left(\frac{\alpha_2}{\alpha_1}\right) - g(x)\right] = 0$$

When we take expectations over x, the third expression in the first order condition integrates to the incentive compatibility condition and drops out, so we are left with

$$1 + \rho\lambda_1 E[\exp(-\rho w(x) - w_0)] = 0$$

Using the same argument that we used in the hidden information model, we can prove that the participation constraint must hold equality, and this fact enables us to solve for λ_1. We can therefore appeal to the participation constraint to solve for λ_1 as:

$$\lambda_1 = -\rho^{-1}\left(\frac{\alpha_2}{\alpha_0}\right)$$

By inspection the first order condition can also be expressed as

$$1 + \rho\lambda_1 \exp[-\rho(w(x) - w_0)]\left[1 + \mu\left(\frac{\alpha_2}{\alpha_1}\right) - \mu g(x)\right] = 0$$

where $\mu \equiv \lambda_2/\lambda_1$ is the ratio of the Lagrange multipliers. Substituting the solution for λ_1 into this equation and multiplying both sides of the resulting equation by $\exp(\rho w(x))$ yields:

$$\exp[\rho(w(x) - w_0)] = \left(\frac{\alpha_2}{\alpha_0}\right)\left[1 + \mu\left(\frac{\alpha_2}{\alpha_1}\right) - \mu g(x)\right]$$

Taking logarithms of both sides and dividing through by ρ, we obtain an equation characterizing the optimal managerial compensation contract for diligent work, which we discuss in the next subsection.

Finally μ itself is fully determined by the other parameters in the model. From the formula for compensation derived above:

$$\left(\frac{\alpha_2}{\alpha_0}\right)\exp[-\rho(w(x) - w_0)] = \left\{1 + \mu\left(\frac{\alpha_2}{\alpha_1}\right) - \mu g(x)\right\}^{-1}$$

Again taking expectations over x on both sides of this equation and appealing to the participation constraint we obtain:

$$1 = E\left\{\left[1 + \mu\left(\frac{\alpha_2}{\alpha_1}\right) - \mu g(x)\right]^{-1}\right\}$$

Depending on the specification of $f_1(x)$ and $f_2(x)$, numerical algorithms are typically required to solve this equation for μ, but for an important class of probability density functions, step functions, it is straightforward to compute μ analytically, as an example below demonstrates.

Optimal Compensation

The discussion above implies:

$$w(x) = w_0 + \rho^{-1}\log\left(\frac{\alpha_2}{\alpha_0}\right) + \rho^{-1}\log\left[1 + \mu\left(\frac{\alpha_2}{\alpha_1}\right) - \mu g(x)\right]$$

Despite its lengthy derivation this equation is remarkably simple to interpret. The benchmark for the manager's compensation comes from his outside employment opportunities w_0, the first term on the right side of the equation. This benchmark is modified by the working conditions in the firm, captured in the second expression. The third expression represents the risk the manager must be exposed to so that his incentives are brought into alignment with the firm's directors.

If $\mu = 0$ then $\lambda_2 = 0$ too, and the incentive compatibility constraint is not binding and the managerial compensation is determined as established in the full information case. In that special case the difference between $w(x)$ and w_0 is $(\rho^{-1}\log\alpha_2 - \rho^{-1}\log\alpha_0)$, the compensating differential required to equalize the value of the change in working conditions from switching from the outside job to diligent employment with the firm. However this solution only applies if $\alpha_2 \leq \alpha_1$ or the signal is fully revealing.

If $\alpha_2 > \alpha_1$ and the signal is not fully revealing then $\mu > 0$. In this case compensation depends on the signal x, and noting that the manager is risk averse, it follows that the expected value of the third expression is positive. If x is a measure of firm performance, such as abnormal return, $g(x)$ is typically a decreasing function throughout much of its range, and hence $w(x)$ increases with superior firm performance.

An Example of Probability Distributions that generate the Signal

For example, suppose that when the manager works diligently, the signal comes from a uniform distribution with lower bound -1 and upper bound 1, and that two steps of equal height and length on the same support comprise the probability density function for the signal when the manager shirks. Mathematically $f_2(x) = 1/2$ on $x \in [-1,1]$, while $f_1(x)$ is $3/4$ on $x \in [-1,0]$ and $1/4$ on $x \in (0,1]$. These assumptions imply $g(x)$ is $3/2$ on $x \in [-1,0]$ and $1/2$ on $x \in (0,1]$.

Because $g(x)$ only takes two values, we see from the compensation equation that $w(x)$ should only take on two values too, a low one for $x \in [-1,0]$ and a high one for $x \in (0,1]$. That is

$$w(x) = w_0 + \rho^{-1}\log\left(\frac{\alpha_2}{\alpha_0}\right) + \begin{cases} \rho^{-1}\log\left[1+\mu(\frac{\alpha_2}{\alpha_1}) - \frac{3\mu}{2}\right] \text{ if } x \in [-1,0] \\ \rho^{-1}\log[1+\mu(\frac{\alpha_2}{\alpha_1}) - \frac{\mu}{2}] \text{ if } x \in (0,1] \end{cases}$$

If x represents abnormal returns this result implies that the manager should be paid the same amount regardless of whether profits are $1/2$ or $2/3,$ but that there is a quantum jump in compensation at the zero profit level. The reason is that the likelihood that the manager worked diligently versus shirked is the same at $x = 1/2$ and $x = 2/3$ whereas $g(x)$ is not continuous at $x = 0.$ To reiterate, the compensation package should reflect the inference that can be made from the signal about the manager's choices; for the purposes the profits of the firm are an ancillary statistic.

To solve for μ we define $\alpha = \alpha_2/\alpha_1$ and substitute the specialization of $g(x)$ and $f_2(x)$ into the solution equation for μ derived above in the general case and obtain:

$$1 = \int_{-1}^{0} \frac{1}{2}\left[1+\mu\alpha-\mu\frac{3}{2}\right]^{-1}dx + \int_{0}^{1} \frac{1}{2}\left[1+\mu\alpha-\mu\frac{1}{2}\right]^{-1}dx$$

Multiplying both sides of the integrated equation by the product of $(2+2\mu\alpha-3\mu)$ and $(2+2\mu\alpha-\mu)$ yields the quadratic form

$$(4\alpha^2 - 8\alpha + 3)\mu^2 + (8\alpha - 4)\mu - 4\alpha = 0$$

The two roots of this equation are

$$\frac{2 \pm 2\sqrt{1+\alpha(3-2\alpha)/(2\alpha-1)}}{(3-2\alpha)}$$

This formula establishes that one root is strictly positive for all $\alpha > 1$ except for the value $\alpha = 3/2.$ The positive root is the solution for μ and it can be substituted back into the compensation equation. At $\alpha = (3+d\alpha)/2$ we see that if $x \in [-1,0]$ then

$$\mu d\alpha = -2 \pm 2\sqrt{1 - \frac{\alpha d\alpha}{(2\alpha-1)}}$$

and

$$w(x) = w_0 + \rho^{-1}\log\left(\frac{\alpha_2}{\alpha_0}\right) + \rho^{-1}\log\left[\sqrt{1-\frac{\alpha d\alpha}{(2\alpha-1)}}\right]$$

thus proving that $w(x)$ takes on the full insurance wage if $\alpha = 3/2$ and $x \in [-1,0].$ In this special case

$$\mu d\alpha = -2 + 2\sqrt{1 - \frac{\alpha d\alpha}{(2\alpha-1)}}$$

the manager earns a bonus of

$$\rho^{-1}\log\left[1+\mu+-1\pm 1\sqrt{1-\frac{\alpha d\alpha}{(2\alpha-1)}}\right]$$
$$= \rho^{-1}\log[1+\mu+d\alpha]$$

Evaluating the importance of moral hazard

There are three ways of measuring the importance of moral hazard, how much the shareholders are willing to pay to eliminate the problem of moral hazard altogether, the benefits accruing to the manager from tending his own interests instead of his shareholders', and the gross loss shareholders would incur (before accounting for managerial compensation) from the manager tending his own interests.

Denote by Δ_1 the amount shareholders would be willing to pay to restructure the job of the manager so that moral hazard is eliminated. Typically it is not practical to do so, but it may be a possibility worth considering when defining some contracting jobs. Our discussion implies that Δ_1 is the risk premium shareholders pay the manager for accepting a job with variable compensation. If there was no moral hazard the firm would pay the manager the fixed wage $\omega_2(u_0)$. Therefore the firms' willingness to pay for eliminating the moral hazard issue is

$$\Delta_1 = E[w^o(x)] - \omega_2(u_0)$$

In the exponential case this simplifes to

$$\rho^{-1}E\left\{\log\left[1+\mu\left(\frac{\alpha_2}{\alpha_1}\right)-\mu g(x)\right]\right\}$$

We denote by Δ_2 the importance of moral hazard from the manager's perspective, which is his value from shirking. Recall $\omega_2(u_0) > \omega_1(u_0)$ because $u_2(w) < u_1(w)$ for all w : in words the manager prefers doing the first activity to the second, and therefore must be compensated more to undertake the latter. The second measure is just the reservation wage difference

$$\Delta_2 = \omega_2(u_0) - \omega_1(u_0)$$

or

$$\rho^{-1}\log\left(\frac{\alpha_2}{\alpha_0}\right)$$

in the exponential case

The loss shareholders would incur from having the manager choose activity a_1 instead of a_2 is denoted by Δ_3, which in this framework is:

$$\begin{aligned}\Delta_3 &\equiv \int_{\underline{x}}^{\infty} f_2(x)[1-g(x)]dx \\ &= E\{f_2(x)[1-g(x)]\}\end{aligned}$$

Which course of action the shareholders should take can be determined by comparing these measures. We suppose that Δ_0 denotes the cost of monitoring the manager and eliminating the moral hazard problem. If $\Delta_0 > \Delta_1$ (as would typically be the case of chief executives), then shareholders should not choose the monitoring option. In the absence of monitoring, the net benefit to shareholders from inducing the manager to work diligently rather than shirk is

$$\int_{\underline{x}}^{\infty} f_2(x)[1-g(x)]dx - [\omega_2(u_0)+\omega_1(u_0)] = \Delta_3 - \Delta_1 + \Delta_2$$

Whether this amount it positive or negative determines whether the shareholders will seek incentives the manager's activities or not. Finally if $\Delta_0 < \Delta_1$ then shareholders might eliminate the moral hazard or give the manager the freedom of discretion in running the firm. In that case we compare

$$\Delta_0 + \int_{\underline{x}}^{\infty} f_2(x)[1-g(x)]dx - \omega_2(u_0) + \omega_1(u_0) = \Delta_3 - \Delta_0 - \Delta_2$$