Lecture 5
Leadership and Reputation

Reputations arise in situations where there is an element of repetition, and also where coordination between players is possible. One definition of leadership is that it facilitates this coordination. This lecture develop these concepts. We analyze repeated games, and games with multiple equilibrium, showing where there might be a role for leadership, and how reputations might be established and maintained.

Reputation

- Reputation is intimately bound up with repetition.
- For example:

1. Firms, both small and large, develop reputations for product quality and after sales service through dealings with successive customers.
2. Retail and Service chains and franchises develop reputations for consistency in their product offerings across different outlets.
3. Individuals also cultivate their reputations through their personal interactions within a community.

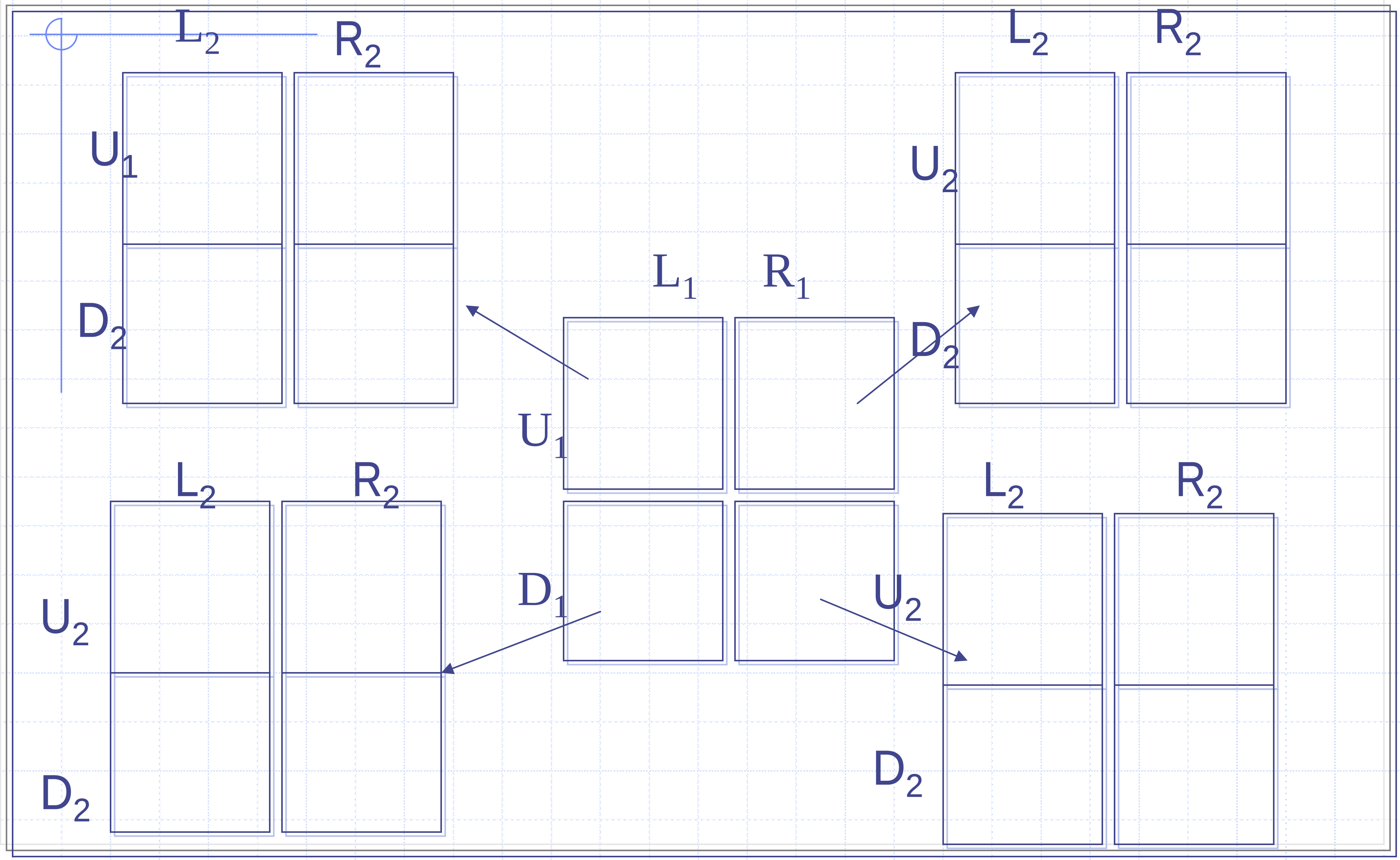
Definition of a repeated game

- These examples motivate why we study reputation by analyzing the solutions of repeated games.
- When a game is played more than once by the same players in the same roles, it is called a repeated game.
- We refer to the original game (that is repeated) as the kernel game.
- The number of rounds count the repetitions of the kernel game.
- A repeated game might last for a fixed number of rounds, or be repeated indefinitely (perhaps ending with a random event).

Games repeated a finite number of times

- We begin the discussion by focusing on games that have a finite number of rounds.
- There are two cases to consider. The kernel game has:
    1. a unique solution
    2. multiple solutions.
- In finitely repeated games this distinction turns out to be the key to discussing what we mean by a reputation.

A 2x2 matrix game played twice



Strategy space for the row player

- Writing down the strategy space for repeated games is a tedious task. In this case the row player must decide in the first period between U_1 and D_1.

- Then for each history the row player must pick between U_2 or D_2. There are 4 cells the player could reach after the first round, and 2 possible instructions to give:

(U_1,L_1)		(U_1,R_1)		(D_1,L_1)		(D_1,R_1)		
2	X	2	x	2	x	2	=	16

- Therefore the row player has a total of 32 pure strategies from which to choose.

Exploiting subgame perfection

- One method for solving this two period game is to write down the 32 by 32 matrix and derive the strategic form solution.
- Another approach is to investigate only subgame perfect equilibrium.
- In this case we would solve the subgame for the final period, and substitute the equilibrium payoffs from the last period into the second last round.
- If there are more than two rounds, we would proceed like that using backwards induction.

A 3 period repeated game

Row firm	Column Firm: Ln	Column Firm: Rn
Un	5 3	4 1
Dn	2 2	1 4

In period n = 1, 2, and 3 the row player picks U_n or D_n and the column player picks L_n or R_n.

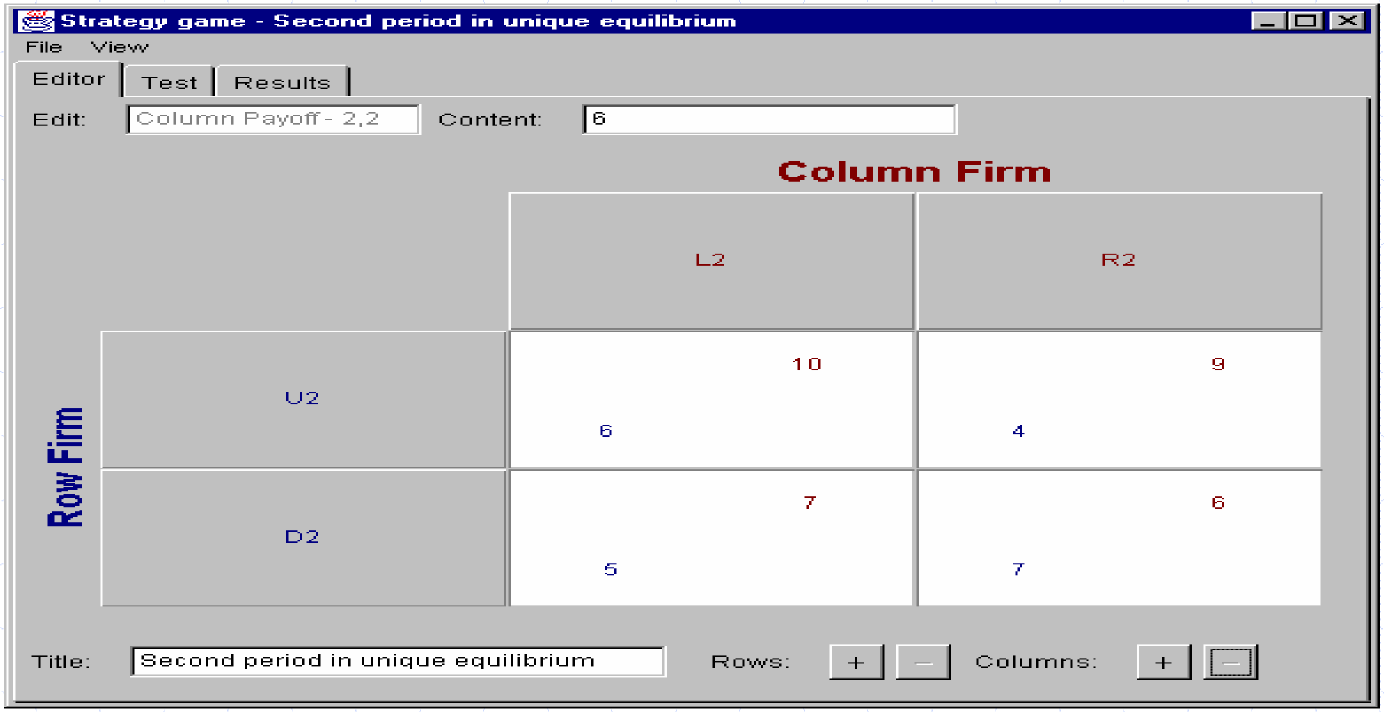
The last period in a finite horizon game

		Column Firm	
		Ln	Rn
Row firm	Un	5 3	4 1
	Dn	2 2	1 4

In this example the unique solution to the n^{th} kernel game is (U_n, L_n), which is found by iterative dominance.

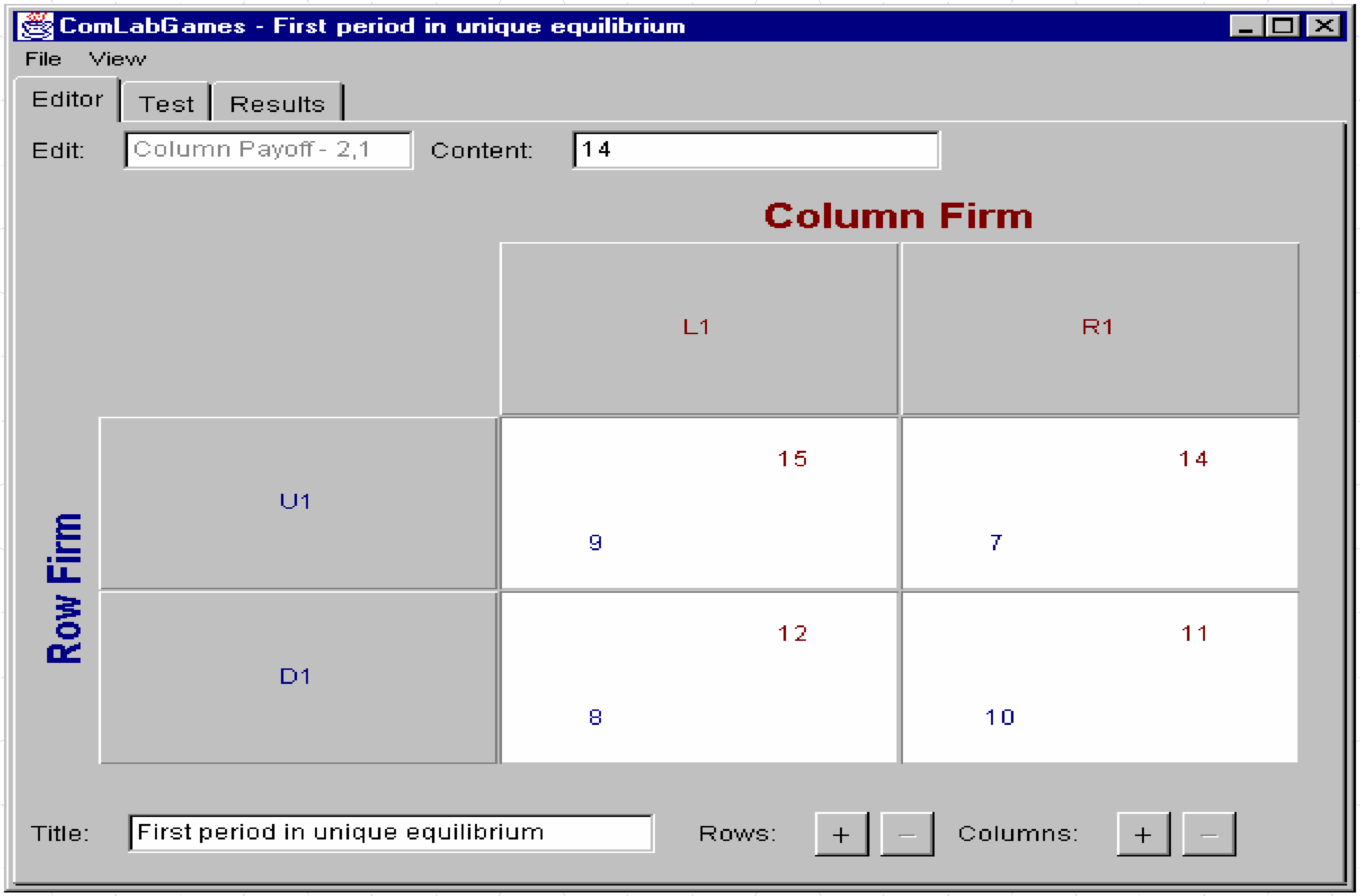
The reduced subgame starting at Period 2

Folding back, the strategic form of the reduced game starting at period 2 is:



The solution to the reduced game is is (U_2, L_2).

The reduced game starting at Period 1



Folding back a second time, the solution is (U_1, L_1).

Solution

- The preceding discussion proves that the unique solution to this three period game is (U_n, L_n) for each n = 1, 2, and 3.
- The reason we obtain a tight characterization of the solution to the repeated game is that the solution to the kernel game is unique.
- Indeed if a game has a unique solution, then repeating the game a finite number of times will simply replicate the solution to the original kernel game.

A theorem on games repeated a finite number of times

- Suppose the kernel of a repeated game has a unique solution. If the repeated game has a finite number of rounds, it has a unique solution, which is to play the solution of the kernel game each round (without regard to prior history).
- This result extends to stage games, games played in sequence by overlapping groups of players. If there is a unique solution to every stage game, and there are only a finite number of stages, the solution to the whole sequence of games is simply found by forming the solutions to all the stages.

Repeated bargaining

- We have in fact already analyzed an example of a repeated game with a unique equilibrium.
- Recall the ultimatum game in which the same party proposed each period.
- Here we found that the unique equilibrium for the one round game of extracting (almost) all the rent carried through to a multi-round setting.

When reputations are irrelevant and leadership is redundant

- These examples and results show that neither reputation nor leadership count when all the kernel games in a finite horizon stage game have a unique equilibrium.
- Reputations and leadership can only arise when at least one of the following two conditions is present:
  1. There are multiple solutions to at least one of the kernel games.
  2. The kernel games are repeated indefinitely.
- The rest of this lecture concentrates on these two situations.

Multiple equilibrium in repeated games with a finite number of rounds

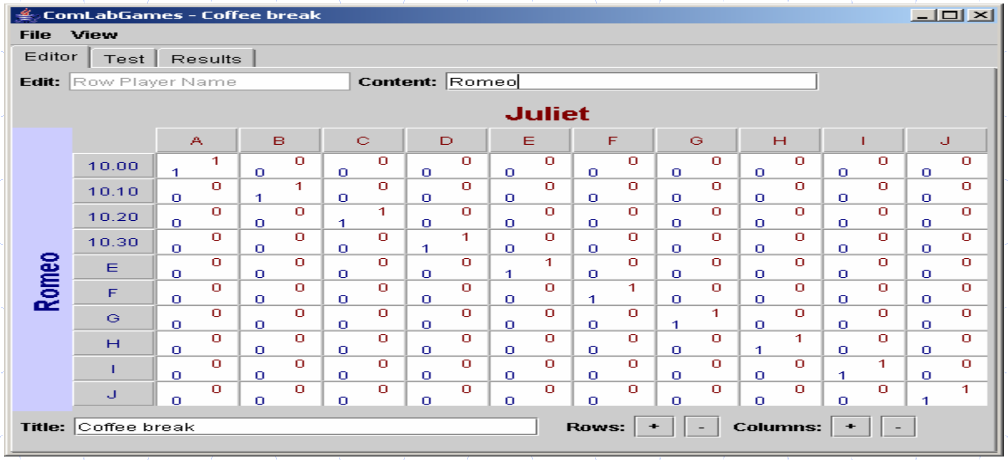
- What happens when there are several equilibrium in the kernel game?
- We will see that the number of solutions in the repeated game increase dramatically.

Coordination games

- In a coordination game there are no conflict of interest between the players. The objectives of the players coincide.
- However there are multiple solutions to the game.
- Unless players coordinate on a specific solution, then they all receive a lower payoff than they would attain if there is coordination.
- What is the probability that players will spontaneously coordinate, and how many iterations does it take before we use mutually compatible strategies to achieve a common goal?

Coffee break

- In the following game if both players take coffee at the same time, then each has an excuse to engage in small talk. Otherwise no meeting takes place.
- The strategic form of the game is illustrated below. There are ten pure strategy equilibrium (and many more mixed strategy equilibrium, all of which achieve lower payoffs).



Romeo \ Juliet	A	B	C	D	E	F	G	H	I	J
10.00	1, 1	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
10.10	0, 0	1, 1	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
10.20	0, 0	0, 0	1, 1	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
10.30	0, 0	0, 0	0, 0	1, 1	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
E	0, 0	0, 0	0, 0	0, 0	1, 1	0, 0	0, 0	0, 0	0, 0	0, 0
F	0, 0	0, 0	0, 0	0, 0	0, 0	1, 1	0, 0	0, 0	0, 0	0, 0
G	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	1, 1	0, 0	0, 0	0, 0
H	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	1, 1	0, 0	0, 0
I	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	1, 1	0, 0
J	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	1, 1

When will a spontaneous meeting occur?

- Furthermore every choice is part of exactly one pure strategy equilibrium. If each player initially chooses a time randomly, then the probability of meeting each other is one tenth.
- If a meeting occurs, we assume the players will coordinate in future by agreeing when to meet. Otherwise we suppose that players pick their coffee breaks as before.
- In that case, a meeting takes place with probability 1/10 on the first day, 9/100 the second day (9/10 times 1/10), 19/1000 the third day and so on.

Arranging meetings

- If there are N players who play an analogous game, an induction argument demonstrates that the probability of them spontaneously meeting together (in a one shot game) is 10^{1-N}.
- Now we change the structure of the game by giving one player, called the leader, power to send a message to the others proposing a meeting time.
- This immediately (and trivially) resolves the coordination problem, and establishes:
  1. the value of coordination to the organization
  2. the potential rent leaders can extract by reducing the coordination that takes place without their active involvement.

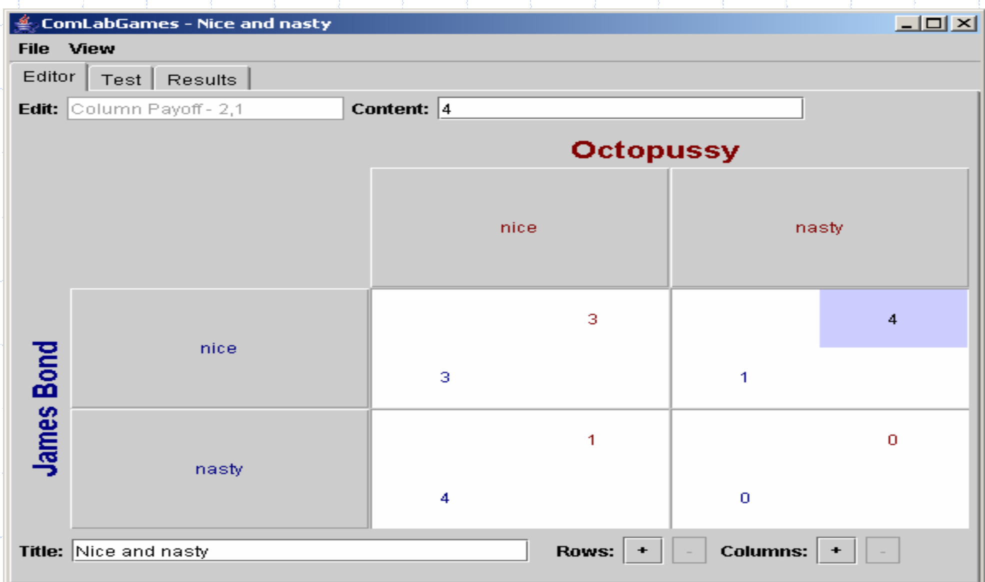
Leadership

- We define a leader as someone who chooses a pure strategy solution in a games where there are multiple pure strategy solutions.
- Note that leaders do not have an enforcement role, since by definition an equilibrium is self enforcing.
- In the examples we have reviewed on meetings, the coordination or leadership function is easy to play. We would not expect anyone to extract rents from performing this role because of competitive pressure to reduce the rent.
- However this need not be the case. Sometimes experience or skill is necessary to recognize potential gains to the players in the game.

Recognizing possibilities for coordination

- In the previous examples it was easy to identify the set of coordinated strategic profiles.

- But they are not always so evident. Let us consider the following example, this time as a finitely repeated game.

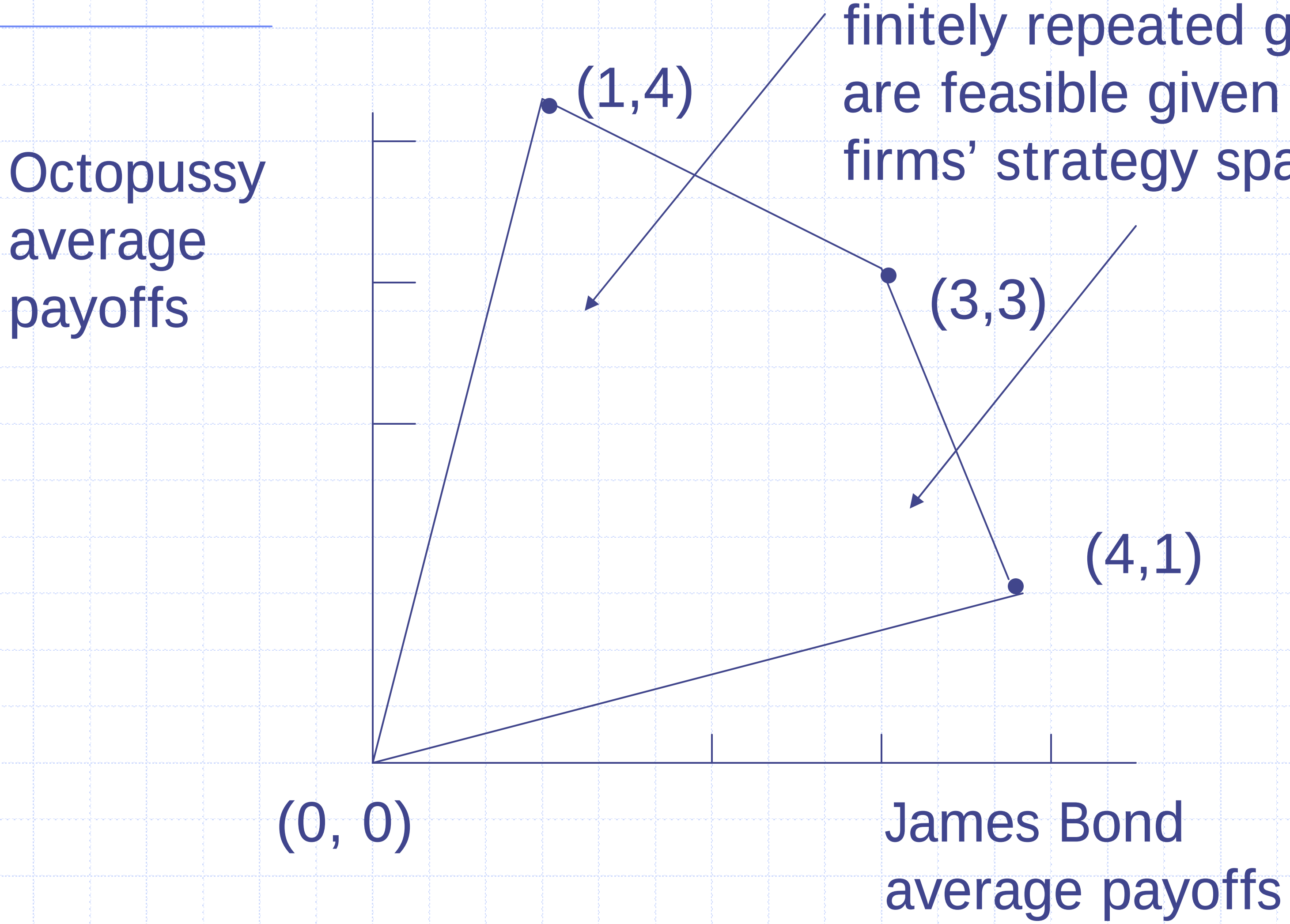


Some subgame perfect equilibrium paths

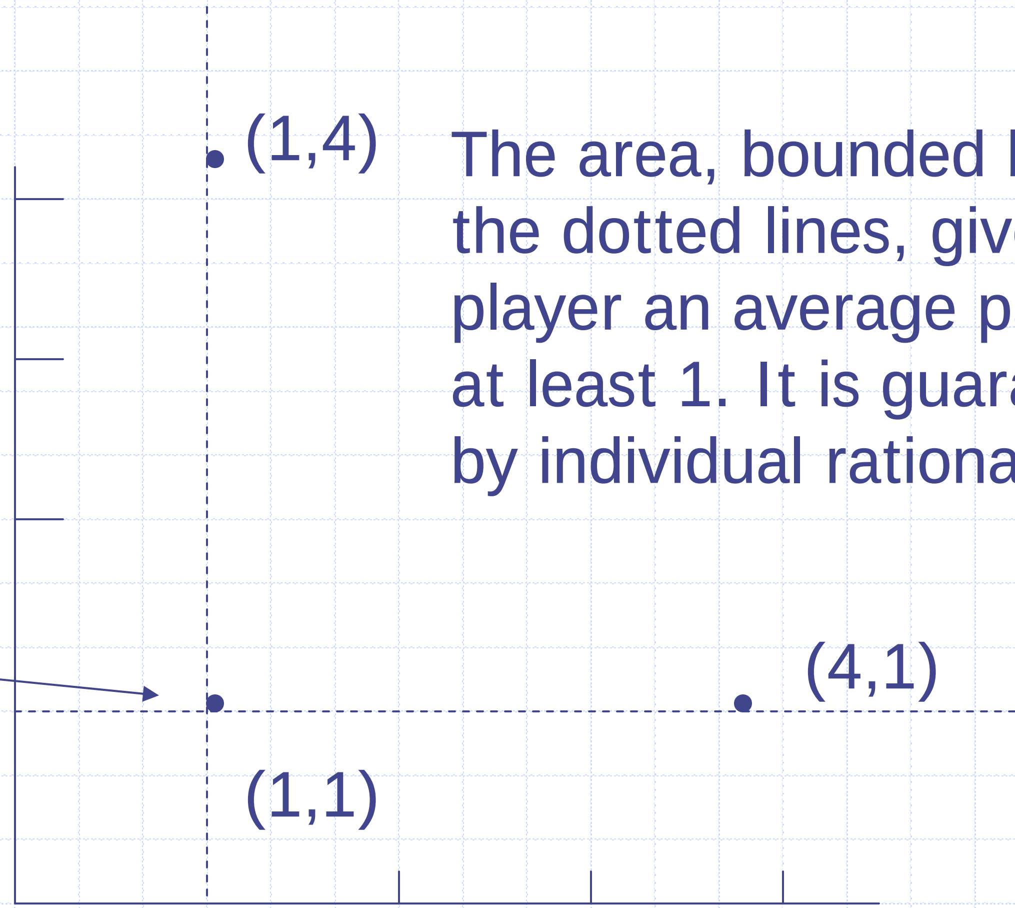
- (4,1), (4, 1)... N times
- (1,4), (1,4)... N times
- (4,1),(1,4)... N times
- (3,3),(1,4),(4,1) (3 rounds)

Feasible average payoffs

This area shows what average payoffs in a finitely repeated game are feasible given the firms' strategy spaces.

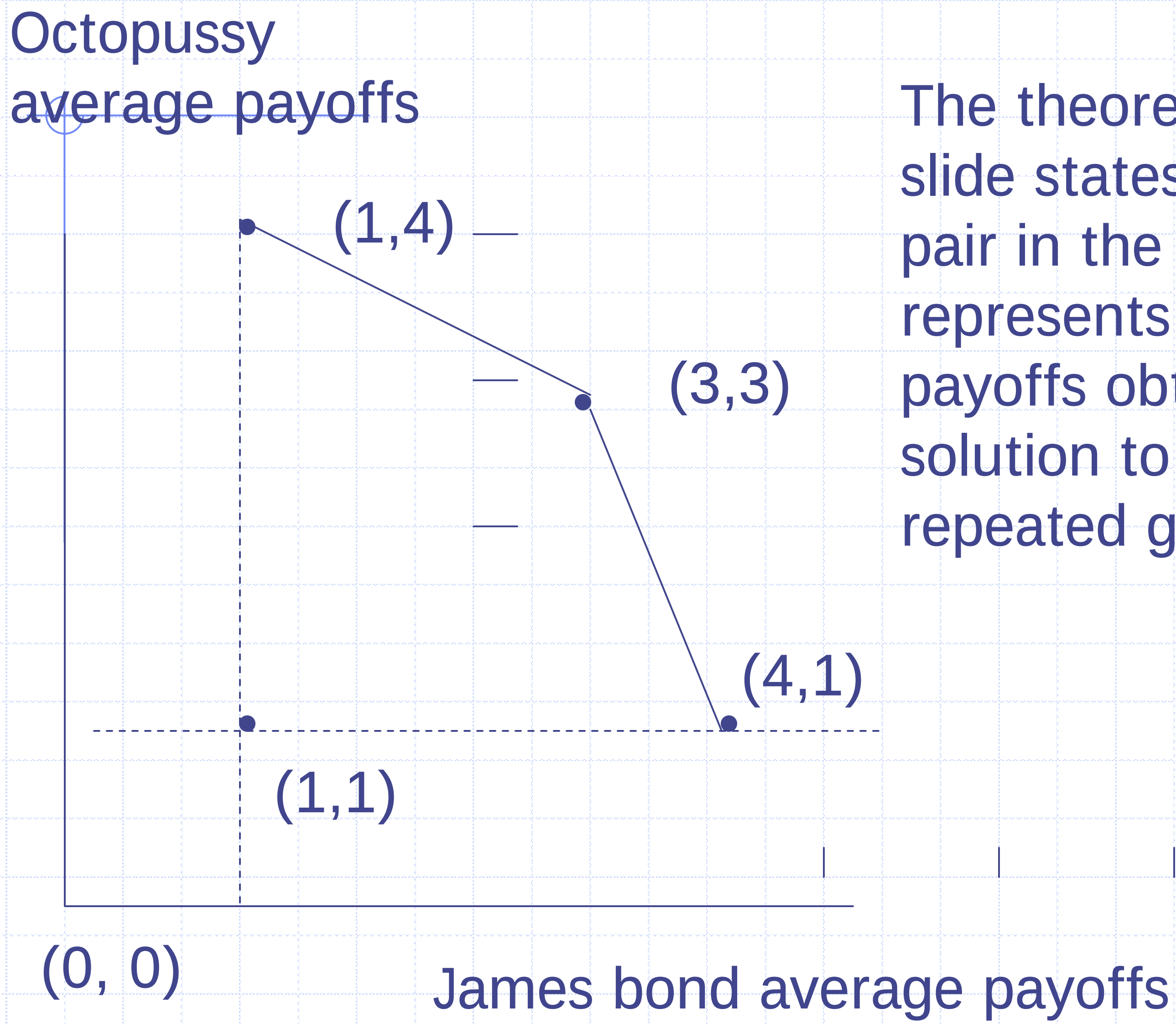


Individual rationality



The area, bounded below by the dotted lines, gives each player an average payoff of at least 1. It is guaranteed by individual rationality.

Average payoffs in equilibrium



The theorem in the next slide states that every pair in the enclosed area represents average payoffs obtained in a solution to the finitely repeated game.

Folk theorem

- Let w_1 be the worst payoff that player 1 receives in a solution to the one period kernel game, let w_2 be the worst payoff that player 2 receives in a solution to the one period kernel game, and define $w = (w_1, w_2)$

- In our example $w = (1,1)$

- Folk theorem for two players: Any point in the feasible set that has payoffs of at least w can be attained as an average payoff to the solution of a repeated game with a finite number of rounds.

Can Bond and Octopussy both earn more than 6 in a three period game?

- The outcome {(3,3), (1,4), (4,1)} comes from playing:

  $\{(\text{nice}_1, \text{nice}_1), (\text{nice}_2, \text{nasty}_2), (\text{nasty}_3, \text{nice}_3)\}$.

- Is this history the outcome of a solution strategy profile to the 3 period repeated game?

Strategy for Bond

Round 1: nice_1

Round 2: $(\ldots, \text{nice}_1) \rightarrow \text{nice}_2$

otherwise $\rightarrow \text{nasty}_2$

Round 3: $(\text{nasty}_1, \ldots) \rightarrow \text{nice}_3$

otherwise $\rightarrow \text{nasty}_3$

Bonds should be nice in the first round. If Octopussy is nice in the first round, Bond should be nice in the second round too. If Octopussy is nasty in the first round, Bond should be nasty in the second. Bond should be nasty in the final round, unless he was nasty in the first round.

Strategy for Octopussy

Round 1: $nice_1$

Round 2: $(\ldots, nasty_1) \rightarrow nice_2$

otherwise $\rightarrow nasty_2$

Round 3: $(nasty_1, \ldots) \rightarrow nasty_3$

otherwise $\rightarrow nice_3$

Octopussy should be nice in the first round. Then if she followed her script in the first round, she should be nasty in the second. However if she forgot her lines in the first round and was nasty, then she should be nice in the second round. If Bond has was nasty in the first round, Octopussy should be nasty in the final round, but nice otherwise.

Verifying this strategy profile is a solution

- Note that the last two periods of play, taken by themselves, are solutions to the kernel game, and therefore strategic form solutions for all sub-games starting in period 2.

- To check whether being nice is a best response for James bond given that Octopussy chooses according to her prescribed strategy we compare:

Checking for deviations by Bond in the first round

Compare

1.	$(\text{nice}_1, \text{nice}_1)$	3
2.	$(\text{nice}_2, \text{nasty}_2)$	1
3.	$(\text{nasty}_3, \text{nice}_3)$	4

		8

with

1.	$(\text{nasty}_1, \text{nice}_1)$	4
2.	$(\text{nice}_2, \text{nasty}_2)$	1
3.	$(\text{nice}_3, \text{nasty}_3)$	1

		6

Since 8 > 6 Bond does not profit from deviating in the first period. A similar result holds for Octopussy.

Therefore, by symmetry, the strategy profile is a SPNE.

Unforgiven

What is the lowest sum of payoffs in the 3 period repeated game that can be supported by a SPNE?

The outcome

$$\{(0,0), (1,4), (4,1)\}$$

is induced by playing

$$\{(\text{nasty}_1, \text{nasty}_1), (\text{nice}_2,\text{nasty}_2), (\text{nasty}_3,\text{nice}_3)\}$$

Can this outcome be supported by a SPNE?

Strategy profiles supporting Unforgiven

Strategy for Clint Eastwood:

Round 1: $nasty_1$

Round 2: $(\ldots, nice_1) \rightarrow nasty_2$ otherwise $\rightarrow nice_2$

Round 3: $(nice_1, \ldots) \rightarrow nice_3$ otherwise $\rightarrow nasty_3$

Strategy for the Sheriff:

Round 1: $nasty_1$

Round 2: $(\ldots, nice_1) \rightarrow nice_2$ otherwise $\rightarrow nasty_2$

Round 3: $(nice_1, \ldots) \rightarrow nasty_3$ otherwise $\rightarrow nice_3$

Checking for a solution

- Using the same methods as before one can show this is also a solution strategy profile for the three period game.

- More generally by punishing any deviation from the equilibrium path with the unfavorable kernel equilibrium repeated until the end of the game guarantees any payoff pair that averages more than the value given by individual rationality.

Results from finitely repeated games

- To summarize:

1. If the kernel game has a unique solution, then the solution to the repeated game is to play the solution of the kernel in each round.
2. If a kernel game for two players has multiple solutions, then the area enclosed by the payoffs and the individual rationality constraints determines the set of average payoffs that can be attained.
3. Leaders choose amongst multiple solutions to achieve coordination between players. The less the potential for coordination between players, the greater the rent that leaders can extract.

Infinite horizon repeated games

- Now we will analyze games that last indefinitely, continuing with some positive probability period after period.
- In this class of repeated games, the horizon is not fixed in advance at a finite number of rounds. Instead the game never ends, or the game ends with some probability after each round.
- We refer to both cases as infinite horizon repeated games.
- If the game lasts forever, payoffs in the future are discounted relative to the present. Otherwise it is hard to define the sum of total payoffs.

An expanded strategy space

- When players realize that their relationship does not have a foreseeable terminal node, new possibilities for cooperation and mutual benefit emerge.
- Cooperative behavior between group members can sometimes be enforced despite their individually conflicting objectives, by threatening to use strategies that punish actions that harm the collective interest.
- In this way we extend the results we found for the principal agent game on rent extraction, where they are multiple solutions to finitely repeated kernel games.

An example showing how play proceeds

		Column Player	
		ht	lt
Row Player	Ht	R R	D d
	Lt	d D	r r

If the game has lasted t rounds, at that time the Row Player picks H_t or L_t, and the Column Player simultaneously picks h_t or l_t. At the end of the period, players accumulate the payoff implied by their collective choices. Then a random variable determines whether play will continue at least one more period.

Discounting the future

- Now consider the following multi-period extension, in which the future payoffs are discounted by:

$$\beta = \frac{1}{1+i}$$

where i is the interest rate.

- The payoffs to the row player are now:

$$\sum_{t=0}^{T} \beta^t u_1(s_{1t}, s_{2t})$$

Unique equilibrium in the Kernel game

Suppose R > r, D > R and r > d.

Row Player	Column Player: ht	Column Player: lt
Ht	R, R	d, D
Lt	D, d	r, r

In the kernel (one period) game there is a unique Nash equilibrium, (L,l) which is dominance solvable.

Finite horizon case

The total payoffs to the column player at the end of the game may be expressed as:

$$\sum_{t=0}^{T} \beta^t u_2(s_{1t}, s_{2t})$$

where:

- there are T rounds or periods in the game;
- s_{1t} is the period t move of the row player (either H or L)
- s_{2t} is the period t move of the column player (either l or r)
- $u_2(s_{1t}, s_{2t})$ is the period t payoff to the column player evaluated in period t currency units
- B is the discount factor that gives the exchange rate between period t payoffs relative to payoffs in period 1

Solving finite horizon games

- The arguments we discussed for finite horizon games extends in a simple way to this class of games.

- If there is a unique solution to the kernel game(s), the solution to a finite round game formed from the kernel game(s) is to sequentially play the unique solution(s) of the composite kernel games.

- In this case the unique solution to this game is:

$$\{(L_t, l_t)\}_{t=0}^{T}$$

Infinite horizon case

- What happens if T = ∞?
- Or equivalently what happens when there is a positive probability at the end of each round that the game will continue one more round?
- As before, one possibility is:

$$\{(L_t, l_t)\}_{t=0}^{T}$$

Are there any other solutions?

- Consider the following strategy profile:

  - For the row player, in period u:

$$\{s_{it}, s_{2t}\}_{t=0}^{u-1} = \begin{Bmatrix} \{(H_t, h_t)\}_{t=0}^{u-1} \text{—>} H_t \\ otherwise \text{—>} L_t \end{Bmatrix}$$

  - For the column player, in period u:

$$\{s_{it}, s_{2t}\}_{t=0}^{u-1} = \begin{Bmatrix} \{(H_t, h_t)\}_{t=0}^{u-1} \text{-->} h_t \\ otherwise \text{-->} l_t \end{Bmatrix}$$

Trigger strategies

- Each player picks the high (collusive) price, unless the player has evidence that either of them have cheated in the past, in which case they pick the low price.
- This is called a *"trigger strategy"* .

Is the trigger strategy profile a solution?

- To determine whether the trigger strategies are a solution, we only need to check whether the sub-games are solved by them.

- There are two kinds of sub-games, depending on whether somebody has cheated in the past or not.

The punishment phase

- Note that regardless of the history up until now the strategy profile:

$$\{s_{it}, s_{2t}\}_{t=u}^{\infty} = \{L_t, l_t\}_{t=u}^{\infty}$$

is subgame perfect.

- It immediately follows that if cheating has occurred at some point in the recent or distant past, it is a the subgame is solved by continuing the punishment phase forever.

The cooperative phase

- All that remains to check is whether whether playing (H_u, h_u) is a best response in period u if nobody has cheated up until now, and the history is

$$\{s_{it}, s_{2t}\}_{t=0}^{u-1} = \{H_t, h_t\}_{t=0}^{u-1}$$

Cooperating

Using the formula for summing a geometric series, that says:

$$\sum_{t=0}^{\infty} \beta^t = 1/(1-\beta)$$

we obtain the value of continuing to cooperate by charging the high price:

$$\sum_{t=u}^{\infty} \beta^t R = R\beta^u/(1-\beta)$$

Defecting

- Now consider the value of defecting by charging a low price in the current period. Since the other player charges a high price the payoff this period is D. But from next period onwards, both players will charge the low price because the punishment phase will begin (and never end). In this case the player gets:

$$\beta^{u} D + \sum_{t=u+1}^{\infty} \beta^{t} r$$

$$= \beta^{u} D + \beta^{u+1} r / (1 - \beta)$$

Which is more profitable?

Taking the difference cooperation can be sustained as a solution to this repeated game if the expression:

$$\beta R^{u} / (1-\beta) - \beta^{u} D - \beta^{u+1} r / (1-\beta)$$

$$= \beta R^{u} + \beta R^{u+1} / (1-\beta) - \beta^{u} D - \beta^{u+1} r / (1-\beta)$$

$$= [(R - D) + (R - r)\beta / (1-\beta)]\beta^{u}$$

is positive.

A further simplification

- Thus cooperation can only occur if the punishment from deviating offsets its immediate gain:

Punishment from deviating

Immediate gain from deviating

$$\frac{\beta(R - r)}{1 - \beta} > D - R$$

or

$$\beta/(1 - \beta) > (D - R)/(R - r)$$

A numerical example

In this case:

$R = 10$

$D = 20$

$r = 5$

Thus:

$D - R = 10$

$R - r = 5$

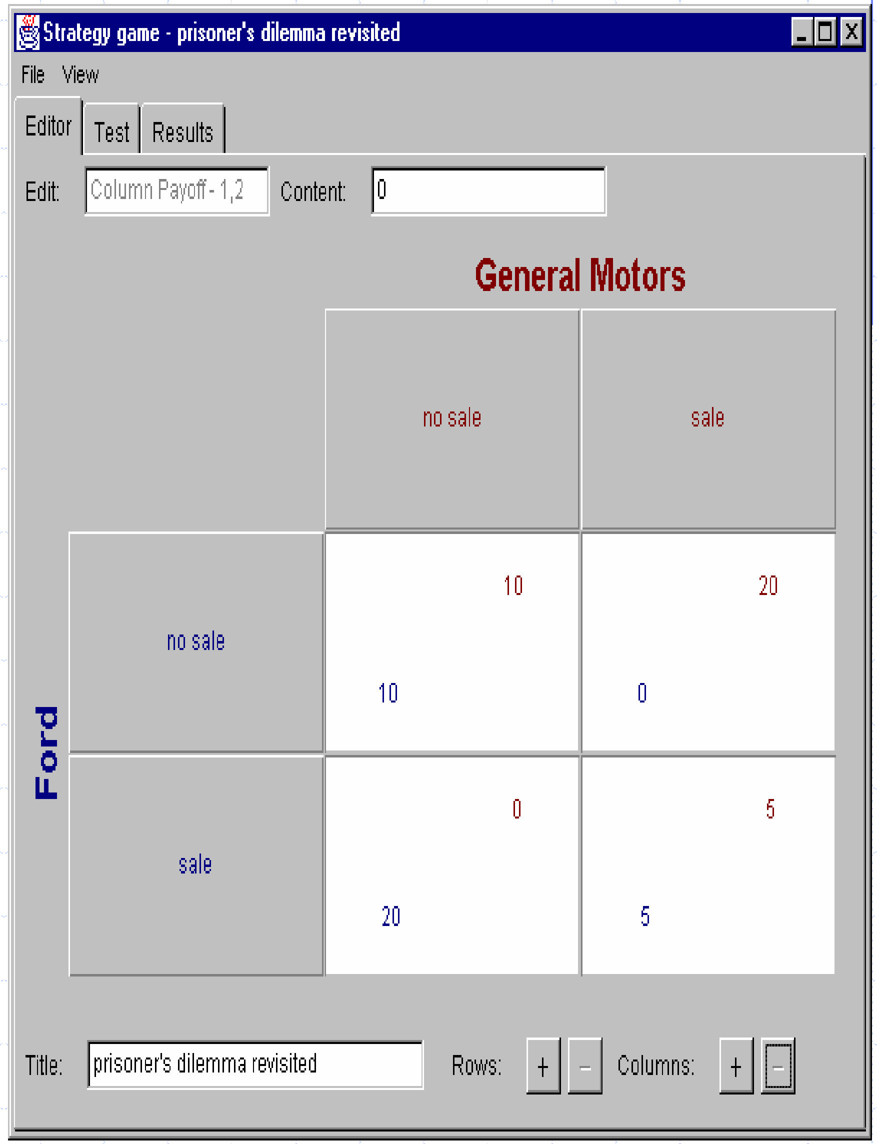
And therefore:

$(D - R)/(R - r) = 2$

So if:

$B/(1-B) > 2$

cooperation at the non-sale price can be sustained as a solution to this repeated game.



Results from infinitely repeated games

- If a kernel game is uniquely solved, there is a unique solution to a game that repeats the kernel a finite number of times. However there may be multiple solutions if the kernel is repeated indefinitely.
- Opportunities for coordination depend on the payoff parameters and the probability of repetition (or the discount factor.)
- In a trigger strategy solution, playing what would be the best reply in the kernel yields less than the long term benefits obtained by cooperating with the other players.
- In trigger strategy solutions, the players jointly engage in this form of strategic investment.