Week 6
Fundamentals of Investment Valuation

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- We have discussed how managers are induced to pursue the objectives of the board.
- But where do those cash streams comes from?
- This final section of the course turns to the underlying choices that determine how projects are evaluated and managed.
- In this sense we are analyzing the value of real options.

- Suppose that the annual sum of the pleasures from retirement plus pecuniary retirement benefits have a monetary value of $b(t)$ where t is the person's age.
- Say $b(t)$ is an increasing function; as people age their retirement benefits increase, as does the effort of continuing to work.
- The current value of earnings from working at age t is denoted by $y(t)$.
- Let $y(t)$ is declining with age, or is a concave function with a maximum.
- Finally assume it is optimal to work when young, so that $y(25) > b(25)$.

- Letting T denote age at death, and R age at retirement, the person chooses R to maximize

$$\int_{25}^{R} y(t)\exp(-rt)dt + \int_{R}^{T} b(t)\exp(-rt)dt$$

- Denoting the optimal age of retirement by R^o, it is straightforward to see that R^o uniquely solves $y(R^o) = b(R^o)$.
- In this case the person works if and only if $y(t) > b(t)$.
- Although this is a dynamic model of optimal stopping, treating it as a static one yields the solution.
- More generally the value of the human capital asset is simply

$$\int_{25}^{R} \max\{y(t), b(t)\}\exp(-rt)dt$$

- How would we incorporate switching costs?

- The current value of a new product to the market is

$$h_0 + h_1\tau$$

  where h_0 is the lifetime earnings of unskilled laborer, and h_1 is the rate at which quality improves with more development.
- We also the development costs from further testing are c per unit of time.
- Therefore the present value of postponing introduction until time τ is

$$\exp(-r\tau)\,(h_0 + h_1\tau)$$

- The costs of development until τ are

$$\int_0^{\tau} c\exp(-r\tau)dt = \frac{c}{r}\left[1 - \exp(-r\tau)\right]$$

- The student chooses τ to maximize her objective function, which is the sum of the human capital component and the nonpecuniary benefit component. The first derivative of this sum with respect to τ is
$$\exp(-r\tau)\left[h_1 - r\left(h_0 + h_1\tau\right) - c\right]$$
We define τ^* as the value of τ which equates this expression to zero and thus uniquely solves first order condition:
$$h_1 - c = r[h_0 + h_1\tau^*]$$
- Each term has an intuitive meaning. Extending the schooling phase by a marginal amount yields net benefits of $h_1 - c$. The interest lost by marginally delaying lifetime career earnings is $r[h_0 + h_1\tau^*]$.
- Solving for τ^* we obtain
$$\tau^* = \left(h_1 - c - rh_0\right)/rh_1$$

- To fully solve this problem we check the boundary conditions.
- Let τ^0 denote the solution to the optimization problem. For all $\tau > \tau^*$ the objective function declines in τ, because the first derivative is negative. Hence $\tau^0 \leq \tau^*$.
- The second derivative of the objective function can be expressed as

$$-r \exp(-r\tau) \left[h_1 - rh_1 \left(\tau - \tau^*\right)\right]$$

- Inspecting the first and second derivatives, the sum of lifetime earnings and net nonpecuniary benefits from schooling is a concave increasing function for all values of positive $\tau < \tau^*$.
- Consequently it is optimal to acquire τ^* schooling if and only if the net benefit from the first unit of schooling is positive, or

$$h_1 - rh_0 - c > 0$$

- Hence $\tau^0 = \tau^*$ if the inequality above is satisfied and $\tau^0 = 0$ otherwise.

3. Machine Replacement

Renewal problem

- The periodic replacement of a durable good is another example of an optimal stopping problem.
- The decision maker chooses when to liquidate an aging asset and begin again.
- For example, suppose the productive asset deteriorates at a constant rate over time, denoted by δ, yielding a service flow worth $q\exp(-\delta t)$ at age t.
- We also assume the scrap value of the old machine is γ, the firm buys a new machine for γ_0, and downtime at the plant from replacing the machine is ρ.
- Given a constant interest rate r, the net present value from buying new machine for γ_0 at date 0, running it from date ρ to date $\tau+\rho$, and then selling it for scrap is:

$$-\gamma_0 + \int_{\rho}^{\rho+\tau} q\exp(-rt-\delta t)dt + \gamma\exp(-r\rho - r\tau)$$

- The optimal policy is to replace the asset at evenly spaced intervals.
- Intuitively, only the age of the existing machinery should affect whether the equipment is now replaced.
- Normalize the machine price γ_0 and the scrap value γ in terms of the service units q, settting $q = 1$.
- The firm's value from installing new machinery every $\tau + c$ periods is

$$\left\{ \int_{\rho}^{\rho+\tau} \exp(-rt - \delta t)dt + \gamma \exp(-r\rho - r\tau) - \gamma_0 \right\} \sum_{s=0}^{\infty} \exp\left[-rs\left(\tau + \rho\right)\right]$$

- Integrating and summing the geometric series, we obtain

$$\frac{\gamma \exp(-r\rho - r\tau) - \gamma_0 + \left(r + \delta\right)^{-1}\left[1 - \exp(-r\tau - \delta\tau)\right]\exp(-r\rho)}{1 - \exp\left[-r\left(\tau + \rho\right)\right]}$$

- The optimal replacement age τ_0 is solved numerically from the first order condition.

4. Project Priorities
Sequential projects

- Dynamic considerations arise from ordering projects that cannot be undertaken simultaneously, but must be undertaken sequentially.
- This constraint might arise because of a scarce factor that might be used on all of them.
- For example a drilling rig might be used to explore one of several oil tracts at a time, each of which is a separate investment project.
- The expertise of a manager might be required by several projects, none of which can be undertaken without her active involvement.
- Suppose an expert is required to supervise two investment projects, called A and B, with net present values of v_A and v_B respectively.
- Project A lasting a periods and Project B for b periods.
- Suppose both projects have positive present value, but each project require the undivided attention of the expert.
- Which one should be undertaken first?

- Project A should be undertaken before Project B if the present value of doing A and then B exceeds the present value of doing B and then A:

$$v_A + \beta^a v_B > v_B + \beta^b v_A$$

- Subtracting $\left(\beta^a v_B + \beta^b v_A\right)$ from both sides of the inequality, and then dividing though by the product $(1 - \beta^a)\left(1 - \beta^b\right)$ gives:

$$\frac{v_A}{1 - \beta^a} > \frac{v_B}{1 - \beta^b}$$

- Noting:

$$\frac{v_A}{1 - \beta^a} = v_A \left(1 + \beta^a + \beta^{2a} + \ldots\right)$$

  it follows that A should be completed first if successively undertaking an infinite sequence of type A projects yields a higher present value than undertaking an infinite sequence of type B projects.

- Another interpretation of the optimal sequencing rule stems from the formula:
$$1-\beta^{a}=(1-\beta)\sum_{s=0}^{a-1}\beta^{s}$$
- Substituting for $1-\beta^{a}$, and also $1-\beta^{b}$, in the inequality determining optimal precedence, A should be undertaken before B if
$$\frac{v_A}{\sum_{s=0}^{a-1}\beta^{s}}>\frac{v_B}{\sum_{s=0}^{b-1}\beta^{s}}$$
- The numerator is the net present value of the project, say in dollars.
- The denominator is the present value of receiving a dollar each period for the project life, the sum of discounted time periods.
- Thus the quotient is the net present value per unit of discounted time.

- This formula for prioritizing two projects that take a known amount of time can be extended in several directions.
- For example if the amount of time is uncertain, and a and b are random variables, then the formula becomes:

$$\frac{v_A}{E\left[\sum_{s=0}^{a-1}\beta^s\right]} > \frac{v_B}{E\left[\sum_{s=0}^{b-1}\beta^s\right]}$$

- Suppose A can undertaken in I stages, denote by a_i the length of the i^{th} stage, and let w_{ai} denote the incremental present value of completing the i^{th} stage.
- Then the present value of partially completing $H \leq I$ stages is

$$v_{AH} = \sum_{i=1}^{H} \beta^{(a_1 + \ldots + a_{i-1})} w_{ai}$$

- Also let B take K stages, where b_k and w_{bk} respectively denote the length and incremental present value of completing the k^{th} stage.
- The index method applies here too, so A should be started before B if

$$\max_{H \leq I} \left\{ \frac{\sum_{i=1}^{H} \beta^{(a_1 + \ldots + a_{i-1})} w_{ai}}{\sum_{i=1}^{H} \beta^{(a_1 + \ldots + a_{i-1})}} \right\} > \max_{J \leq K} \left\{ \frac{\sum_{k=1}^{J} \beta^{(b_1 + \ldots + b_{k-1})} w_{bk}}{\sum_{k=1}^{J} \beta^{(b_1 + \ldots + b_{k-1})}} \right\}$$

- Having partially completed A, and not yet started B, the same logic can be used to determine A should be continued or not, by reformulating the remaining stages of A as a new project.

- The framework we have used to study dynamic decisions can readily adapted to analyze spacial location problems too.
- Suppose a cooperative food retailer buys its product from various farms scattered throughout the region, and must determine the number, size and location of its suppliers.
- There are a countable number of potential farm suppliers, which we index by the distance from the central market t. The cost of growing output x at location t is the quadratic function

$$\alpha_{0t} + \alpha_{1t}x + \alpha_{2t}x^2$$

where $\mu_{0t} < 0$ represents a fixed cost that must be incurred maintain the farm, $\mu_{1t} > 0$ is a linear coefficient in food output, and $\mu_{2t} < 0$ is the coefficient on the quadratic term representing declining marginal productivity that sets in as less fertile areas of the farm are cultivated.
- Aside from cultivation costs, there are transportation costs, denoted by γ_t, and spoilage, which we denote by δ_t.

- Let $1\{x_t > 0\}$ indicate whether the farm joins the cooperative or not.
- The coop procurement manager chooses the sequence $\{x_t\}_{t=0}^{\infty}$ that minimizes its total costs

$$\sum_{t=0}^{\infty} \left(1\{x_t > 0\}\alpha_0 + \alpha_1 x_t + \alpha_2 x_t^2 + \gamma_t x_t\right)$$

subject to demand from the marketing department's requirement that quantity x be displayed on the shelves for sale

$$\sum_{t=0}^{\infty} (1 - \delta_t) x_t \geqslant x$$

- The Lagrangian for the optimization problem is

$$L = \sum_{t=0}^{\infty} \left[1\{x_t > 0\}\alpha_0 + \alpha_1 x_t + \alpha_2 x_t^2 + \gamma_t x_t - \lambda_0 (x_t - \delta_t x_t - x) + \lambda\right.$$

where λ_0 is the shadow price of relaxing the demand constraint imposed by marketing, and the $\{\lambda_t\}_{t=1}^{\infty}$ is the sequence of Lagrange multipliers ensuring non-negativity from each farm supplier.

- The first order conditions for this problem are:

$$\alpha_1 + 2\alpha_2 x_t + \gamma_t - \lambda_0 \left(1 - \delta_t\right) + \lambda_t = 0$$

which generates the interior solution

$$x_t^o = \frac{\lambda_0 \left(1 - \delta_t\right)}{2\alpha_2} - \gamma_t - \alpha_1$$

- Using the fact that it is not optimal to buy more produce than the amount that satisfies the marketing departments requirements we solve for λ_0 to obtain

$$\textstyle\sum_{t=0}^{\infty} \left(1 - \delta_t\right) \left[\frac{\lambda_0 \left(1 - \delta_t\right)}{2\alpha_2} - \gamma_t - \alpha_1\right] = x$$

and solve for x_t.

6. Resource Extraction

Undeveloped resources

- Depletable resources such as oil and minerals are typically used in production or consumed soon after they have been extracted.
- They are less costly to store in their natural state than artificially, extraction costs are delayed if the resource is not immediately required, and transportation costs between the extraction site and the final consumer destination market are lower if there is no need to temporarily store quantities of the resource in another location.
- Consequently firms roughly determine the current consumption of natural resources and the amount preserved for future periods through their mining and oil drilling activities.
- Real estate development has similar features. You should only develop an area as much as demand for potential buyers will support.

- Suppose there is a fixed stock q of the exhaustible resource that is depleted by extraction and consumption.
- As a function of its current price in period t, which we denote by p_t, the quantity demanded is p_t^{η}, where η, a negative constant, is the constant elasticity of demand.
- Hence revenue in period t is $p_t^{1+\eta}$ and the overall resource constraint facing the firm is captured by the inequality

$$\sum_{t=0}^{\infty} p_t^{\eta} \leq q$$

- If there are no extraction costs of the resource, the value of the firm is the net present value of revenue

$$\sum_{t=0}^{\infty} b_t p_t^{1+\eta}$$

where b_t is the price of a t period bond bought at date $t = 0$ when the firm makes its drilling plans.

- Thus the owner of the resource chooses a sequence of quantities $\{q_t\}_{t=1}^{\infty}$ to maximize the Lagrangian

$$L = \sum_{t=0}^{\infty} b_t p_t^{1+\eta} + \lambda \left(q - \sum_{t=0}^{\infty} p_t^{\eta} \right)$$

where λ_0 is the shadow value of a new discovery.

- The first order condition for p_t is

$$(1+\eta)\, b_t p_t = \lambda \eta$$

- Note that $b_t p_t$, the present value of an extracted unit, is equalized across all periods.

- Substituting for quantity from the demand equation into the FOC and making q_t the subject of the resulting equation yields

$$q_t = \left[\frac{\lambda \eta}{(1+\eta)\, b_t} \right]^{\eta}$$

- All the resources are extracted, otherwise an additional unit would have no value. This implies

$$q = \sum_{t=0}^{\infty} q_t = \sum_{t=0}^{\infty} \left[\frac{\lambda \eta}{(1+\eta)\, b_t} \right]^{\eta}$$

- Solving for λ, the value of discovering another resource unit, completes the solution

$$\lambda = q^{\frac{1}{\eta}} \left\{ \sum_{t=0}^{\infty} \left[\frac{\eta}{(1+\eta)\, b_t} \right]^{\eta} \right\}^{-\frac{1}{\eta}}$$

- Substituting the solution for q_t into the net present value equation gives back the value of the field.

7. Internally Funded Investment

Retained earnings

- Why are retained earnings an important source of investment capital?
- One explanation is technological. We might suppose that raw materials for investment (say a new variety of bird seed) can only be siphoned off from output, which would otherwise be sold.
- Another more common explanation for internal growth is when there is incomplete information, firms seeking to carve out niches and create new markets find it more difficult to borrow against their future visions than firms making investments with known returns.
- On the one hand they have to convince their potential creditors that their ideas are well founded to show that they are realistic and not fraudulent.
- On the other hand if large sums of credit are required to fund a successful enterprise if even larger profits are anticipated in the future.
- Unless the entrepreneur has some other way of protecting the rent from his idea the more he shares it with others he exposes it, and hence reduces its value.

- We consider the life cycle of product of a firm which produces revenue from its plant of size k_t at period t. Given capital of k_t in period t, it produces material of k_t^γ which can be retailed as a consumption good, denoted by x_t, or transformed into capital for next period $t+1$. The internal financing constraint is then

$$k_t^\gamma = x_t + k_{t+1}$$

- The period t current revenue from selling the output x_t is $\log(x_t)$.
- Given an initial start up capital of k_0 the firm chooses plant size $\{k_t\}_{t=1}^T$, or equivalently retail output $\{x_t\}_{t=1}^T$ to maximize its present value

$$\sum_{t=0}^T (1+r)^{-t} \log(x_t)$$

subject to the internal financing constraint.

7. Internally Funded Investment

A recursion

- To solve this problem we substitute for x_t in the firm's objective function using the internal financing constraint to obtain

$$\sum_{t=0}^{T} \beta^t \log\left(k_t^\gamma - k_{t+1}\right)$$

- In current value terms for period t, the marginal loss from an extra investment unit from foregone sales is $\left(k_t^\gamma - k_{t+1}\right)^{-1}$, while the marginal value from converting the investment unit to sales next period is the product of marginal revenue from sales next period $(1+r)^{-1}\left(k_{t+1}^\gamma - k_{t+2}\right)^{-1}$ and the increase in production attributed to the extra unit of investment $\gamma k_{t+1}^{\gamma-1}$.
- The first order condition for an interior solution equates these two values:

$$\left(k_t^\gamma - k_{t+1}\right)\gamma\beta k_{t+1}^{\gamma-1} = \left(k_{t+1}^\gamma - k_{t+2}\right)$$

  This second order difference equation can be simplified with a change in variables.

7. Internally Funded Investment

Solution to recursion

- Defining $K_{t+1} \equiv k_{t+1} / k_t^{\gamma}$ as the ratio of new capital on current output the first order condition can be rewritten as:

$$\gamma\beta\left(1 - K_t\right) = \left(1 - K_{t+1}\right) K_t$$

- A solution to this first order difference equation is:

$$K_t = \gamma\beta \frac{1 - \left(\gamma\beta\right)^{T-t+1}}{1 - \left(\gamma\beta\right)^{T-t+2}}$$

which can be confirmed by substituting the proposed solution for $\left(1 - K_t\right)$ and K_t into the difference equation to obtain:

$$\gamma\beta\left(1 - K_t\right) = \gamma\beta \frac{1 - \gamma\beta}{1 - \left(\gamma\beta\right)^{T-t+2}} = \left(1 - K_{t+1}\right) K_t$$

- Moreover, the firm should not undertake any investment in the last period because the marginal revenue from sales is positive at every level of production, whereas there is no scrap value from retaining capital beyond period T. Therefore $k_{T+1} = K_{T+1} = 0$.

- In the market for perishable goods and personal services, balancing demand flow against supply orders and deliveries affects the profitability of the enterprise.
- Consider the problem of restocking items when demand is perfectly predictable.
- Suppose demand for the item is constant each period, but the store faces nonlinear reordering costs.
- If it orders $c_t \in \{1, 2, \ldots\}$ in period t, it pays a fixed cost of α_0 for the order plus unit costs of α_1.
- Scaling the quantity units by the amount demanded each period, and denominating unit values by the item's sale price, the net profits in period t are:

$$u_t = 1 - 1\{x_t > 0\}\alpha_0 - \alpha_1 x_t$$

- The store maximizes it value by minimizing the present value of total supply costs subject to the constraint that it demand is met each period.
- Let s_t denote store inventory at period t. It follows the law of motion:

$$s_{t+1} = s_t - 1 + c_{t+1}$$

- Thus the store minimizes

$$\sum_{t=0}^{\infty} \beta^t \left[1\left\{x_t > 0\right\} \alpha_0 + \alpha_1 x_t\right]$$

  subject to the constraint that $s_t \geq 0$.
- These assumptions ensure that the optimal policy is to order the same quantity of the item at evenly spaced intervals.
- Intuitively, the problem facing the firm only depends on the current inventory, not the inventory policy that has been pursued in the past. It is also easy to see that the firm would not order anything until its inventories are exhausted.

- These observations considerably simplify the task of deriving the optimal inventory policy. Instead of choosing c_t each period as a function of s_t, the firm simply orders an optimally determined s every s periods to minimize:

$$\sum_{t=0}^{\infty} \beta^{st} \left(\alpha_0 + \alpha_1 s\right) = \left(1 - \beta^s\right)^{-1} \left(\alpha_0 + \alpha_1 s\right)$$

The first order condition for this problem is

$$\alpha_1 \left(1 - \beta^s\right)^{-1} + \left(\alpha_0 + \alpha_1 s\right) \left(1 - \beta^s\right)^{-2} \beta^s \log \beta = 0$$

Simplifying we obtain

$$\left(1 - \beta^s\right) \alpha_1 + \left(\alpha_0 + \alpha_1 s\right) \beta^s \log \beta = 0$$

the for demanders in the face of uncertain demand, demand flow technology d_t is demand each period with $d_t \leq s_t$ which is current shelf inventory.

- Thus stocks depreciate at rate η implying the law of motion for food inventory is

- On time delivery versus finished product inventory, downtime due to retooling different product lines, reputation for prompt service.
- Suppose there is a backlog of orders for two types of production (x_{0t}, x_{1t}). There is an order flow for each type of good, which must be processed in the order its was received.
- Orders processed early incur no penalty but carry an inventory cost. Orders processed late incur a cost that increases with tardiness.
- Denote by (d_{0t}, d_{1t}) the new orders demanded in period t and (s_{0t}, s_{1t}) the stock of outstanding orders. Managers decide how many of the new orders to accept. Let (g_{0t}, g_{1t}) denote the number of new orders management accepts in period t, where $g_{it} \leq d_{it}$ for each $t \in \{1, 2, \ldots\}$ and $i \in \{0, 1\}$.
- Demand is generated by a probability distribution given by $F(d_{it})$. Production in period t is (y_{1t}, y_{2t}). In period t the plant incurs costs $f(s_{0t}, s_{1t})$ from departures from just-in-time delivery and also setup costs of k if it shifts from one incurred Every time we switch from one

- The state variables for this problem at time t are (s_{0t}, s_{1t}, h_t) where h_t indicates the status of the factory floor:

$$h_t = \begin{cases} 0 \text{ if the factory is set up for producing } x_0 \text{ at time } t \\ 1 \text{ if the factory is set up for producing } x_1 \text{ at time } t \end{cases}$$

  Production at time t is therefore

$$x_{0t} = h_t d_t (1 - \alpha_0) + (1 - h_t)(1 - d_t)$$
$$x_{1t} = h_t (1 - d_t) + (1 - h_t) d_t (1 - \alpha_1)$$

  while the demand flow is

$$s_{0t} = s_{0,t-1} + d_t - x_{0t}$$

  and the current revenue is, where $x_{0t} \leq d_{0t} + s_{0t}$,

$$p_{0t} x_{0t} - [\gamma 1 \{s_{0t} > 0\} + \delta 1 \{s_{0t} < 0\}] s_{0t}^2$$

- If occurs then the plant can process one unit of output Longer runs increase productions reduce the backlog but also the means for

- Often information must be acquired through experience, rather than directly purchased.
- Consider a department within an organization that makes offers to a succession of job candidates.
- Suppose the value of filling the position to the department is a positive number denoted x, and that the department can offer the job to at most to N candidates.
- Moreover if an offer is rejected there is some probability β that the department will lose the position.
- No bargaining takes place between the department and any candidate: the department simply makes one offer to the current candidate which is either accepted or rejected.

- We denote the department's n^{th} offer by w_n.
- If the n^{th} candidate accepts the department's offer then the net benefit to the department is $x - w_n$.
- Each job candidate $n \in \{1, \ldots, N\}$ has a reservation wage which we denote by v_n, meaning that he will accept any wage offer of at least v_n and reject every wage offer below it.
- The department does not observe v_n, but believes the probability distribution of v_n is independently and identically distributed with probability distribution function $F(v)$ over all candidates $n \in \{1, \ldots, N\}$.

- In the last period there is no reason to offer more than $\overline{v}$ since every candidate would accept $\overline{v}$.
- Setting $N = 1$ the expected value to the department from making an offer of w_1 is the probability of acceptance $F(w_1)$ multiplied by the net value conditional on acceptance $(x - w_1)$, or

$$F(w_1)(x - w_1)$$

- Assuming an interior solution, the unconstrained first order condition to this problem is:

$$F(w_1) = (x - w_1) F'(w_1)$$

- Reducing the wage offer by one dollar reduces the expected hiring costs by $F(w_1)$, but the probability of hiring anybody falls by $F'(w_1)$ in which case the firm loses its surplus of $(x - w_1)$.

10. Filling a Vacancy

An induction

- Generalizing this problem to all finite positive integers N and $n \in \{1, \ldots, N-1\}$, let V_n denote the value of the firm when $N - n$ job candidates have rejected their respective offers and the firm follows an optimal wage offer policy for the n remaining candidates.
- Setting $V_0 = 0$, the firm's value follows the recursion:

$$V_n = \max_{w_n} \left\{ (x - w_n) F(w_n) + [1 - F(w_n)] \beta V_{n-1} \right\}$$

- Substituting V_0 and V_1 in the equation above, noting $V_1 \geq 0$,and comparing it to the same equation when V_1 and V_2, we deduce $V_2 \geq V_1$, and can show by the induction principles that $V_n \geq V_{n-1}$ for all $n \leq N$.
- In words, the longer the list of prospective job candidates, the greater the value of the firm.

10. Filling a Vacancy

First order condition for reservation offer

- Differentiating the maximand with respect to w_n we obtain the first order condition:

$$\left(x - w_n^o - \beta V_{n-1}\right) F'\left(w_n^o\right) - F\left(w_n^o\right) = 0$$

- If the stationary point solves the problem it must satisfy the second order condition

$$SOC\left(w_n^o, V_{n-1}\right) \equiv \left(x - w_n^o - \beta V_{n-1}\right) F''\left(w_n^o\right) - 2F'\left(w_n^o\right) < 0$$

- Dropping subscripts and superscripts, and totally differentiating with respect to V and w yields:

$$\frac{dw}{dV} = \frac{\beta F'\left(w_n^o\right)}{SOC\left(w_n^o, V_{n-1}\right)} < 0$$

- The more candidates remaining, the lower the offer, wage offers rising as the number of prospective candidates shrink. With each successive rejection, the expected value of the firm falls for two reasons.

- The probability that the vacancy will not be filled at all increases, and if someone is hired, it will be on less favorable terms to the firm. If the opportunities to hire are unlimited and N is infinite, the value function has the recursive structure

$$V_{\infty} = F\left(w_{\infty}^{o}\right)\left(x - w_{\infty}^{o}\right) + \left[1 - F\left(w_{\infty}^{o}\right)\right]\beta V_{\infty}$$

- Solving for V_{∞} in terms of w_{∞}^{o} yields

$$V_{\infty} = \frac{F\left(w_{\infty}^{o}\right)}{1 - \beta\left[1 - F\left(w_{\infty}^{o}\right)\right]}\left(x - w_{\infty}^{o}\right)$$

- Substituting this equation back into the first order equation yields an implicit solution for w_{∞}^{o}:

$$w_{\infty}^{o} = x - \frac{F\left(w_{\infty}^{o}\right)}{F'\left(w_{\infty}^{o}\right)}\left(1 + \frac{\beta}{1 - \beta}F\left(w_{\infty}^{o}\right)\right)$$

- As an illustrative example if v_n is uniformly distributed on $\left[\underline{v}, \overline{v}\right]$, then $\underline{v} \leq v_n \leq \overline{v}$ with $F(v) = (v - \underline{v}) / (\overline{v} - \underline{v})$, then for $N = 1$ the department chooses w_1 to maximize:

$$(x - w_1) F(w_1) = (x - w_1)(w_1 - \underline{v}) / (\overline{v} - \underline{v})$$

- In the unconstrained maximization problem there is a stationary point at

$$w_1^o = (x + \underline{v})/2$$

- Noting that the division is assured of hiring the candidate if it offers the candidate $\overline{v}$, the optimal offer is therefore $\overline{v}$ if $2x > 2\overline{v} - \underline{v}$ and w_1^o otherwise.
- In this example we assume that $x > \underline{v}$; otherwise the department would have no interest in hiring anyone.
- For $\underline{v} = 0$ and $\underline{v} = 1$ the interior solution apples with $w_1^o = 1/2$.

- In this special case the first order condition for $N > 1$ simplifies to

$$x - w_n^o - \beta V_{n-1} = w_n^o$$

and the recursive equation for the value function is

$$V_n = \left(x - w_n^o\right) w_n^o + \left(1 - w_n^o\right) \beta V_{n-1}$$

- Substituting into the equation for V_n and V_{n-1} from the first order condition we thus obtain the solution for w_n^o in recursive form

$$w_{n+1}^o = \frac{(1-\beta) x + 2\beta w_n^o - \beta \left(w_n^o\right)^2}{2}$$

- Solving for the unique positive root of the quadratic from the formula for derived for w_∞^o implies:

$$w_\infty^o = x - w_\infty^o \left(1 + \frac{\beta}{1-\beta} w_\infty^o\right) = \frac{1-\beta}{\beta}\left[-1 + \sqrt{1 + \frac{\beta}{1-\beta}}\right]$$