Lecture 3 in Competitive Equilibrium
Portfolio Choices

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Demand for financial assets

Asset Pricing

- Individuals and households hold wealth in financial securities to defer consumption.
- For example parents save for the education of their children, individuals save for retirement, and the wealthy bequest future generations with their largesse.
- Half of the value of the stock market is held by a very small fraction of individuals. Nevertheless more than 50 percent of households hold financial securities of some form or other.
- Collectively, these groups, including foreign investors, create the demand for financial securities.

- The main financial securities are:
  1. Stocks, bonds and their derivatives issued by corporations and private enterprises to finance their operations.
  2. Mortgage backed securities that bundle loans on housing stock.
  3. Bonds issued by governments (local, state and federal) to help finance their public expenditures.
  4. Fiat money or currency, and foreign exchange issued national governments, and managed by the banking system.

- If traders only cared about the mean return of an asset, it is hard to justify why assets could have different mean returns.
- There is abundant evidence that assets have different mean returns, suggesting that traders care about other moments of the probability distribution apart from the first.
- For example, suppose that traders are risk averse, rather than risk neutral. In this case they would seek to diversify their portfolio.
- Starting with the basic model of inter-temporal consumption smoothing, we derive the fundamental equation that determines how financial assets are priced in competitive equilibrium.

- Consider the lifetime utility of an individual with preferences:

$$\sum_{t=1}^{T} \beta^t u(c_t)$$

where t is the period or year, β is the subjecitve discount factor, and c_t is consumption in time period t.

- For example the constant relative risk aversion utility function is defined by:

$$u(c_t) = c_t^{\alpha}$$

- As $\alpha \to 0$ this utility function converges to (natural) log utility:

$$u(c_t) = \log(c_t)$$

- As $\alpha \to 1$ it converges to the risk neutral preferences:

$$u(c_t) = c_t$$

If $\alpha < 0$ then utility is decreasing in consumption, and if $\alpha > 1$ then a consumer who maximizes expected utility is a risk lover.

- Suppose there are J financial securities, and let $q_{t-1,j}$ denote the amount of the j^{th} security owned at the beginning of period t.
- Let r_{tj} denote the return on the j^{th} security revealed at the beginning of period t and applying to the securities that were bought in the previous period, and let p_{tj} denote the price of (buying or selling) the j^{th} security in period t in terms consumption units.
- When the trader enters period t with assets $(q_{t-1,1}, \ldots q_{t-1,J})$, he takes the return on them, consumes some, trades some, and holds the rest. Letting c_{tj} denote the amount of consumption (or dividend) extracted from the j^{th} security, the budget constraint is:

$$\sum_{j=1}^{J} p_{tj} \left(r_{tj} q_{t-1,j} - q_{tj} - c_{tj} \right) \geq 0$$

- Let $E_t\left[\cdot\right]$ denote the expectations operator based on information at time t.
- Given her her endowment $\left(q_{t-1,1}, \ldots q_{t-1,J}\right)$ at time t, the consumer maximizes:

$$u\left(c_t\right) + E_t\left[\sum_{s=t+1}^{T} \beta^{s-t} u\left(c_s\right)\right]$$

by consuming

$$c_s = \sum_{j=1}^{J} c_{sj}$$

and making her portfolio choices $\left(q_{s1}, ..., q_{sJ}\right)$ for all $s \in \{t, \ldots, T\}$ subject to the sequence of all the future budget constraints.

- If $u(c_t)$ is strictly increasing, all wealth is consumed, all the budget constraints are met with strict equality, yielding the expression for consumption:

$$c_t = \sum_{j=1}^{J} c_{tj} = \sum_{j=1}^{J} \left(r_{tj} q_{t-1,j} - q_{tj} \right)$$

- This implies we can express for consumption into the objective function, and reformulate the consumer investor's problem as sequentially choosing the vector of assets to maximize:

$$u \left(\sum_{j=1}^{J} \left(r_{tj} q_{t-1,j} - q_{tj} \right) \right) + E_t \left[\sum_{s=t+1}^{T} \beta^{s-t} u \left(\sum_{j=1}^{J} \left(r_{sj} q_{s-1,j} - q_{sj} \right) \right) \right.$$

- The interior first order condition, found by differentiating the objective function with respect to each asset k requires that for each $k \in \{1, \ldots, J\}$ ::

$$u'\left(\sum_{j=1}^{J}\left(r_{tj}q_{t-1,j} - q_{tj}\right)\right) = \beta E_t\left[r_{t+1,k}u'\left(\sum_{j=1}^{J}\left(r_{t+1,j}q_{tj} - q_{t+1,j}\right)\right)\right]$$

- Substituting the definition of consumption back into the first order condition we obtain:

$$u'(c_t) = \beta E_t\left[r_{t+1,k}u'(c_{t+1})\right]$$

or

$$E_t\left[r_{t+1,k}\frac{\beta u'(c_{t+1})}{u'(c_t)}\right] \equiv E_t\left[r_{t+1,k}m_{t+1}\right] = 1$$

where m_{t+1} is the marginal rate of substitution between c_t and c_{t+1} and $\pi_{t+1,k}$ is the real return on the k^{th} security.

- In words, the return on an asset return is discounted by the marginal rate of substitution between current and future consumption.

- When $u(c_t) = c_t^{\alpha}$ the asset pricing equation yields

$$\beta E_t \left[r_{t+1,k} \left(\frac{c_{t+1}}{c_t} \right)^{\alpha-1} \right] = r_{t+1} \beta E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{\alpha-1} \right] = 1$$

- In the log case $u(c_t) = \ln(c_t)$ we obtain

$$\beta E_t \left[r_{t+1,k} \left(\frac{c_t}{c_{t+1}} \right) \right] = r_{t+1} \beta E_t \left[\frac{c_t}{c_{t+1}} \right] = 1$$

- In the linear case the expected return is equated with the interest rate for all interior choices:

$$\beta E_t \left[r_{t+1,k} \right] = \beta r_{t+1} = 1$$

- Since $u(c)$ is a concave increasing function it follows that if no units of the j^{th} asset is held, that is $q_{tj} = 0$ then:

$$u'(c_t) \geq \beta E_t \left[r_{t+1,k} u'(c_{t+1}) \right]$$

- This equation shows the condition under which the distribution of returns on the j^{th} asset are too low to warrant keeping any units.
- If both the j^{th} and k^{th} assets are consumed in period t, then their prices must be equated since they are perfect substitutes in current consumption. In this case $p_{tj} = p_{tk}$
- If the j^{th} asset is consumed in period t, but the k^{th} asset is not, then $p_{tj} < p_{tk}$.

Market clearing

Asset Pricing

- For each trader a first order condition applies to each positively consumed asset; otherwise it is not held.
- These conditions imply there is a solution to the asset allocation of each trader as a function of his asset endowment at the beginning of the period and the joint probability distribution governing asset returns.
- To express the market clearing conditions, we let $\{1, \ldots, N\}$ denote the population of traders, and temporarily superscript asset endowments $q_{t-1,j}^{(n)}$ and allocations $r_{t+1,j} q_{tj}^{(n)}$ to designate trader $n \in \{1, \ldots, N\}$ the population of traders.
- Market clearing in competitive equilibrium means that every period the total amount of assets consumed plus the amount invested exactly offsets its supply:

$$\sum_{n=1}^{N} \left(r_{t,j} q_{t-1,j}^{(n)} - q_{tj}^{(n)} \right) = 0$$

- If the consumer holds a (real) bond, its risk free return

$$r_{t+1} \equiv 1 + i_t$$

where i_t is called the interest rate, must satisfy the equation too:

- $$E_t \left[r_{t+1} m_{t+1} \right] \equiv (1 + i_t) E_t \left[m_{t+1} \right] = 1$$

- That is

$$i_t = \frac{1}{E_t \left[m_{t+1} \right]} - 1$$

- Recall from the definition of a covariance:

$$\begin{aligned} cov\left(r_{t+1,k}, m_{t+1}\right) &= E_t\left[r_{t+1,k} m_{t+1}\right] - E_t\left[r_{t+1,k}\right] E_t\left[m_{t+1}\right] \\ &= 1 - E_t\left[r_{t+1,k}\right] E_t\left[m_{t+1}\right] \end{aligned}$$

  the second line following from the asset pricing equation.
- Dividing both sides of the equation by the expected value of the marginal rate of substitution and rearranging yields

$$E_t\left[r_{t+1,k}\right] - 1/E_t\left[m_{t+1}\right] = -cov\left(r_{t+1,k}, m_{t+1}\right)/E_t\left[m_{t+1}\right]$$

- Noting $cov\left(r_{t+1}, m_{t+1}\right) = E_t\left[r_{t+1} m_{t+1}\right] = 0$ we recall the formula we derived for the risk free rate as

$$r_{t+1} = 1/E_t\left[m_{t+1}\right]$$

- Substituting this equation into the one above it we now obtain the risk correction for the mean return on the jth asset

$$E_t\left[r_{t+1,k}\right] - r_{t+1} = -r_{t+1} cov\left(r_{t+1,k}, m_{t+1}\right)$$

- From the risk correction formula for the mean return on the jth asset, we can write:

$$E_t\left[r_{t+1,k}\right]-r_{t+1}=-r_{t+1}\sigma_{jt}\sigma_{mt}\frac{cov\left(r_{t+1,k},m_{t+1}\right)}{\sigma_{jt}\sigma_{mt}}\equiv -r_{t+1}\sigma_{jt}\sigma_{mt}\rho_{jt}$$

  where ρ_{jt} is the correlation coefficient between m_{t+1} and $r_{t+1,k}$, σ_{jt}^2 is the variance of rjt, and σ_{mt}^2 is the variance of m_{t+1}.
- Hence

$$\frac{E_t\left[r_{t+1,k}\right]-r_{t+1}}{r_{t+1}}=-\sigma_{jt}\sigma_{mt}\rho_{jt}$$

- Since the absolute value of every correlation coefficient is bounded by one, the following inequality must be satisfied in a competitive equilibrium:

$$\left|\frac{E_t\left[r_{t+1,k}\right]-r_{t+1}}{r_{t+1}}\right|\leq\sigma_{jt}\sigma_{mt}$$

  for each asset held by the trader.