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Estimating the Gains from Trade in Limit-Order Markets

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ABSTRACT

We present a method to estimate the gains from trade in limit-order markets and provide empirical evidence that the limit-order market is a good market design. Using observations on order submissions and execution and cancellation histories, we estimate both the distribution of traders' unobserved valuations for the stock and latent trader arrival rates. We use the resulting estimates to compute the current gains from trade, the gains from trade in a perfectly liquid market, and the gains from trade with a monopoly liquidity supplier. The current gains are 90% of the maximum gains and 150% of the monopolist gains.

THE MAJORITY OF THE WORLD'S STOCK EXCHANGES operate some form of a limit-order market. A feature of good market design is that traders realize most of the potential gains from trade. We develop a method for identifying and estimating the gains from trade in a limit-order market and apply our method to a sample from a particular limit-order market, the Vancouver Stock Exchange (VSE). We find that gains from trade in the VSE are approximately 90% of the gains from trade in a perfectly liquid market and approximately 50% more than gains from trade in a market in which liquidity is supplied by a profit-maximizing monopolist. Our results therefore provide new empirical evidence that the limit-order market is a good market design.

A large number of experimental studies document that the gains from trade in a double auction are close to the maximum gains from trade. See, for example, Cason and Friedman (1996) or the survey by Holt (1995). Similarly, our empirical results show that the limit-order market—a market design similar to

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that of the double auction—is remarkably efficient. To our knowledge, our empirical estimates of the gains from trade are the first such estimates computed using field data from a limit-order market.

Our sample comes from the audit tapes of the VSE and contains traders' order submissions and execution histories. The VSE is organized as a limit order market with a market design very similar to that of the London Stock Exchange, Euronext, the Stockholm Stock Exchange, the Toronto Stock Exchange, and the Archipelago Exchange. The VSE is primarily a venture-capital exchange and around two-thirds of its stock listings are in the natural resource industry. It is similar in its objectives to AIM in London, the Second Market and New Market components of Euronext, Nya Marknaden in Stockholm, and the Tokyo Stock Exchange's Mother system (High-Growth and Emerging Stock Markets). All of these markets focus on emerging companies that wish to have access to public equity financing but that do not meet the listing requirements of more senior exchanges.

The companies trading on these venture exchanges have characteristics that differ from a typical New York Stock Exchange stock. They tend to be smaller and less frequently traded, for example. However, it is just these types of differences that make the VSE a particularly appropriate context within which to examine the gains from trade. Investors will care most about the performance of the trading system where liquidity is low and highly variable. In turn, stocks with highly variable liquidity will have large variation in the gains from trade. Studying such stocks gives us a good chance of being able to identify and explain variations in the gains from trade.

Our model is an extension of the model of traders' optimal order submission in limit-order markets in Hollifield, Miller, and Sandås (2004). In that model, the traders' order submissions depend on their valuations for the stock and the trade-offs across execution probabilities, picking-off risks, and order prices for alternative order submissions. Hollifield et al. (2004) derive and test restrictions implied by the model using a semiparametric test and their empirical evidence lends some support to the model framework. Here, we extend their model to continuous time and include an order-execution cost. The extension to continuous time allows our model to deal with the selection problem that arises when some traders find it optimal not to submit an order, and including a more flexible cost structure allows our model to deal with the rejections reported in Hollifield et al. (2004).

The model we use captures a key feature of limit-order markets, namely, the ability of traders to choose whether to submit market or limit orders. In that sense, our model is consistent with the theoretical models of traders' order submissions in limit-order markets of Foucault (1999), Foucault, Kadan, and Kandel (2005), Parlour (1998), and Wald and Horrigan (2005). However, the additional flexibility of our model makes it suitable for empirical work. For example, the traders in our model may choose from multiple limit-order prices, unlike the traders in Parlour (1998); the traders' limit orders in our model may last for multiple periods, unlike the limit orders in Foucault (1999); and limit orders in our model may be canceled, unlike the limit orders in Foucault et al. (2005).

A number of researchers have also developed theoretical models abstracting from the choice between market and limit orders. Glosten (1994) shows that in a competitive environment the limit-order market provides enough liquidity to discourage entry by other competing market designs. Sandås (2001) tests and rejects the restrictions implied by one version of the Glosten (1994) model with discrete prices and a time priority rule as in Seppi (1997). Biais, Martimort, and Rochet (2000) study imperfect competition among a finite number of traders that submit limit-order schedules and show that the limit-order book of Glosten (1994) obtains when the number of limit-order submitters becomes large. Glosten (2006) relaxes the assumption of perfect competition, and shows that under imperfect competition, the limit order market is an optimal market design considering the gains from trade of both traders who submit market orders and traders who submit limit orders.

Our model differs from Biais et al. (2000), Glosten (1994, 2006), and Seppi (1997) in two respects. First, we allow traders to choose between market and limit orders, and second, we model the dynamics of individual order submissions. We do not allow for endogenous market-order quantities or asymmetric information, but we do allow limit orders to face picking-off risk. As in Glosten (2006), we consider the gains from trade that accrue to both traders who submit market orders and traders who submit limit orders. Our empirical evidence on the efficiency of the limit-order market complements Glosten's (2006) theoretical results on the efficiency of the limit-order market.

Many studies document that the empirical frequency of limit- and market-order submissions changes with market conditions. Using a sample from the Paris Bourse, Biais, Hillion, and Spatt (1995) show that traders are more likely to submit limit orders in markets with wide spreads or thin limit-order books. Griffiths et al. (2000) report similar findings for the Toronto Stock Exchange, and Ranaldo (2004) reports similar findings for the Swiss Stock Exchange. Using a sample from the New York Stock Exchange, Harris and Hasbrouck (1996) show that traders are more likely to submit limit orders when the expected payoffs from submitting limit orders increase. We extend the literature by using the empirical variation in the frequency of limit and market order submissions to empirically link traders' order submissions to their valuations. Without an empirical link between traders' valuations and their order submissions, it is not possible to estimate the gains from trade.

Our sample contains both the traders' order submissions and the execution and cancellation histories of the traders' order submissions. In our model, traders' gains depend on their valuations for the stock. We apply a discrete choice model that links the traders' valuations to the traders' observable order submissions using the expected payoffs from the different order submissions that the traders could have made. We use the discrete choice model to estimate the distribution of the traders' valuations, the expected payoffs from alternative order submissions, and the traders' optimal order submission strategy. We use the resulting estimates to compute the gains from trade in the limit-order market, the maximum gains from trade, and the monopoly gains from trade.

The maximum gains from trade are the gains from trade that the traders would obtain in a perfectly liquid market, and the monopoly gains from trade are the gains from trade that the traders would obtain if liquidity were supplied by a profit-maximizing monopolist. The maximum gains from trade provide an upper bound on the gains from trade in any mechanism and a natural benchmark against which to measure the efficiency of a market's design. In our sample of stocks from a relatively illiquid exchange, the gains from trade in the limit-order market are approximately 90% of the maximum gains from trade and approximately 150% of the monopoly gains from trade. In this respect, the limit-order market is a good market design—traders in the Vancouver Stock Exchange realize most of the potential gains from trade.

Section I describes the theoretical model and Section II describes the current, maximum, and monopolist gains from trade in the theoretical model. Section III describes our sample and reports estimates of the model parameters. Section IV reports our estimates of the current, maximum, and monopolist gains from trade and Section V concludes. The appendices contain the proofs and additional technical material.

I. Model

Our model captures several key features of trading in a limit-order market. Any trader can submit market and limit orders, and all traders face the same order submission and order execution rules—there are no designated market makers or other traders with special quoting obligations or trading privileges. Moreover, the market is transparent—all traders observe the limit-order book and general market conditions when making their order submission decisions. Finally, all trades involve a limit order being executed by a market order, with limit orders executed according to strict price and time priority.

A. Model Structure

The model is set in continuous time. Traders arrive sequentially and differ in their valuations of the stock. The finite-dimensional vector x_t denotes the exogenous state variables that determine the conditional trader arrival rate and the conditional distribution of the traders' valuations. The exogenous state variables follow a stationary Markov process.

The probability that a trader arrives between t and $t + dt$ is given by

$$\Pr(\text{Trader arrives in } [t, t + dt) \mid x_t) = \lambda(t; x_t)\, dt, \tag{1}$$

which can be interpreted as

$$\lim_{\Delta t \downarrow 0} \frac{\Pr(\text{Trader arrives in } [t, t + \Delta t) \mid x_t)}{\Delta t} = \lambda(t; x_t). \tag{2}$$

Each trader is risk neutral with valuation of the stock of v_t. We decompose v_t as follows:

$$v_t = y_t + u_t. \tag{3}$$

The random variable y_t is the common value of the stock at t; y_t may be interpreted as the traders' common time t expectation of the liquidation value of the stock. Innovations in the common value are drawn from a stationary process, with possibly time-varying conditional moments.

The random variable u_t is a trader's private value for the stock. Different traders have different private values, creating the opportunity for gains from trade. The private value is an independent random variable drawn from the conditional distribution

$$\Pr(u_t \leq u \mid x_t) \equiv G(u \mid x_t), \tag{4}$$

with density $g(\cdot \mid x_t)$. Once a trader arrives at the market, his private value remains fixed while he has an order outstanding.

The conditional trader arrival rates, the conditional distributions of the innovations in the common value, and the conditional distributions of the private values all depend on the exogenous state variables. The exogenous state variables therefore determine the intensity of trader arrivals, the distribution of changes in the stock's common value, and the aggregate willingness of traders to pay a price that deviates from the common value in order to obtain immediate order execution. An example of such a state variable that we have in mind is lagged common value volatility. For example, following a period of high common value volatility the intensity of trader arrival, the future common value volatility, and the traders' aggregate willingness to pay a price that deviates from the common value for immediate order execution may all change.

A trader who arrives at t has a single opportunity to submit either a market order or a limit order for q shares. We normalize q to one unit. While we can allow the order quantities to vary exogenously across traders in our model, we present the case of unit quantity to reduce notation. In our empirical work, we condition on the observed order quantities. By assuming that each trader has only one order submission opportunity, we abstract from a trader's endogenous future cancellation and resubmission decisions.

The trader's order submission at t depends on the common value, y_t, his private value, u_t, and his information. The trader's information is captured by the exogenous state variables, x_t, and a finite-dimensional vector of endogenous state variables, w_t. Let $z_t \equiv (x_t, w_t)$ denote the state vector that represents the trader's information. The state vector z_t follows a stationary Markov process.

The exogenous state variables x_t predict the future trader arrival rates, the distribution of innovations in the common value, and the distribution of future traders' valuations; the exogenous state variables may therefore predict the execution probabilities and picking-off risks (i.e., the expected loss from such executions) of a new order submission.

The endogenous state vector w_t includes information about the current limit-order book and past order submission activity. For example, the current bid-ask spread is likely to predict the execution probability for limit orders and so the bid-ask spread is an element of w_t. Similarly, a limit order submitted with few limit orders in the book is likely to have a different execution probability and picking-off risk than a limit order submitted with many limit orders in the book. The endogenous state vector w_t therefore includes the current limit-order book.

The endogenous state vector w_t also includes information that is useful in predicting the distribution of cancellations for the orders in the book. Limit orders are executed according to price and time priority. As a consequence, the execution probability of a newly submitted limit order depends on the cancellation probabilities of the existing limit orders in the book. Suppose that the conditional probability that a limit order is canceled depends on how long the limit order has been in the book. In this case, the average age of limit orders in the book helps predict the probability that current limit orders in the book are canceled in the future. For example, past order submission activity is correlated with the age of the unexecuted orders in the book, and so past order submission activity is useful to a trader in predicting the execution probabilities of a new limit order submission.

The decision indicators $d_{t,s}^{sell} \in \{0, 1\}$ for $s = 0, 1, \ldots, S$ and $d_{t,b}^{buy} \in \{0, 1\}$ for $b = 0, 1, \ldots, B$ denote the trader's order submission at t, where s and b index the finite set of available order submissions: $S < \infty$ and $B < \infty$. Let $p_{t,s}^{sell}$ denote the sell price associated with $d_{t,s}^{sell}$, and let $p_{t,b}^{buy}$ denote the buy price associated with $d_{t,b}^{buy}$. If the trader submits a sell market order, then the order price is the best bid quote, $p_{t,0}^{sell}$, and $d_{t,0}^{sell} = 1$. If the trader submits a sell limit order at the price $p_{t,s}^{sell}$, s ticks above the current best bid quote, then $d_{t,s}^{sell} = 1$. Similar definitions apply to the buy side. If the trader does not submit any order at time t, then $d_{t,s}^{sell} = 0$ for all s and $d_{t,b}^{buy} = 0$ for all b.

A limit order is either executed or canceled. We define two latent random times for each order: The latent cancellation time, $t + \tau_{cancel}$, and the latent execution time, $t + \tau_{execute}$. The order is executed at $t + \tau_{execute}$ if $\tau_{execute} \leq \tau_{cancel}$ and the order is canceled at $t + \tau_{cancel}$ if $\tau_{execute} > \tau_{cancel}$. Orders do not last longer than $T < \infty$, thus the random variable τ_{cancel} is bounded from above by $T < \infty$. The distributions of latent times describe the uncertainty about the limit order's outcome.

There is an order submission cost of $c_o \geq 0$ for all types of order submissions. There is also an order execution cost of $c_e \geq 0$: The trader pays a cost of c_e when the order executes. The costs, c_o and c_e, do not depend on the trader's valuation, nor on the trader's order submission at t. One interpretation of c_e is that it represents the commission on the trade. With $c_e = 0$ the payoff from order submissions at t are the same as in Hollifield et al. (2004).

Suppose that a trader with valuation $v_t = y_t + u_t$ submits a buy limit order b ticks below the ask quote at price $p_{t,b}^{buy}$: $d_{t,b}^{buy} = 1$. The conditional distribution of the latent cancellation time depends on the state vector, z_t, and on the order submission itself, but it does not depend on the trader's private value. Conditional on z_t, the latent cancellation time is independent of all other random variables in the model. One interpretation of the conditional independence assumption is that traders find it too costly to continuously monitor their limit orders. The probability distribution of the latent cancellation time is

$$\Pr\left(t + \tau_{cancel} \leq t + \tau \,\middle|\, z_t, d_{t,b}^{buy} = 1\right) = F_{cancel}\left(\tau \,\middle|\, z_t, d_{t,b}^{buy} = 1\right). \tag{5}$$

The conditional distribution of the latent execution time depends on the state vector, z_t, and on the order submission, $d_{t,b}^{buy} = 1$, but not on the trader's private value. The probability distribution of the latent execution time is

$$\Pr\big(t + \tau_{execute} \leq t + \tau \,\big|\, z_t, d_{t,b}^{buy} = 1\big) = F_{execute}\big(\tau \,\big|\, z_t, d_{t,b}^{buy} = 1\big). \tag{6}$$

The execution time depends on the trader's order submission, future order cancellations, and the arrival of future traders and their order submissions. The execution time therefore depends on how future traders behave given their valuations and the order books and information they face, and the latent execution times depend on future traders' order submissions.

The execution time distribution is defined conditional on the current trader's order submission. The conditional distribution of the execution time is therefore different for different order submissions that the trader could make. By conditioning on potential order submissions, the trader considers the feedback from his current order submission to future traders' order submission decisions.

Define the indicator function for order execution as

$$I_t(\tau_{execute} \leq \tau_{cancel}) = \begin{cases} 1, & \text{if } t + \tau_{execute} \leq t + \tau_{cancel}, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The realized utility from submitting a buy order at price $p_{t,b}^{buy}$ is

$$\begin{aligned} & I_t(\tau_{execute} \leq \tau_{cancel})\big(y_{t+\tau_{execute}} + u_t - p_{t,b}^{buy} - c_e\big) - c_o \\ & \quad = I_t(\tau_{execute} \leq \tau_{cancel})\big(y_t + u_t - p_{t,b}^{buy} - c_e\big) \\ & \qquad + I_t(\tau_{execute} \leq \tau_{cancel})(y_{t+\tau_{execute}} - y_t) - c_o. \end{aligned} \tag{8}$$

The first term on the first line is the indicator for execution multiplied by the payoff at execution and the second term is the order submission cost.

Define

$$\psi_b^{buy}(z_t) \equiv E\big[I_t(\tau_{execute} \leq \tau_{cancel}) \,\big|\, z_t, d_{t,b}^{buy} = 1\big] \tag{9}$$

as the execution probability for the order. For a market order, the execution probability is one.

An order may execute after there has been a change in the stock's common value; we refer the expected loss from such executions as the picking-off risk. Define

$$\xi_b^{buy}(z_t) \equiv E\big[I_t(\tau_{execute} \leq \tau_{cancel})(y_{t+\tau_{execute}} - y_t) \,\big|\, z_t, d_{t,b}^{buy} = 1\big] \tag{10}$$

as the picking-off risk for the order. Since a market order executes immediately, the picking-off risk for a market order is zero. Using the law of iterated expectations, the picking-off risk simplifies to

$$\xi_b^{buy}(z_t) = E\big[(y_{t+\tau_{execute}} - y_t) \,\big|\, I_t(\tau_{execute} \leq \tau_{cancel}) = 1, z_t, d_{t,b}^{buy} = 1\big]\psi_b^{buy}(z_t). \tag{11}$$

The picking-off risk is the expected change in the common value between the time of the order submission and the time of the order execution, conditional on execution, multiplied by the probability that the order executes.

The conditional distribution of the latent cancellation times, the conditional distribution of the latent execution times, and the expected change in the common value conditional on execution all depend on the state vector, z_t. As a consequence, the execution probabilities and picking-off risks also depend on the state vector.

The trader's expected utility from submitting a buy order at price $p_{t,b}^{buy}$ is the expected value of equation (8), conditional on the trader's information, which, using the definitions of the execution probability and picking-off risk, is

$$U_b^{buy}(y_t + u_t; z_t) = \psi_b^{buy}(z_t)\big(y_t + u_t - p_{t,b}^{buy} - c_e\big) + \xi_b^{buy}(z_t) - c_o. \qquad (12)$$

Similarly, the expected utility of submitting a sell order at $p_{t,s}^{sell}$ is

$$U_s^{sell}(y_t + u_t; z_t) = \psi_s^{sell}(z_t)\big(p_{t,s}^{sell} - y_t - u_t - c_e\big) - \xi_s^{sell}(z_t) - c_o. \qquad (13)$$

The trader's order submission strategy maximizes his expected utility,

$$\max_{\{d_{t,s}^{sell}\},\{d_{t,b}^{buy}\}} \sum_{s=0}^{S} d_{t,s}^{sell} U_s^{sell}(y_t + u_t; z_t) + \sum_{b=0}^{B} d_{t,b}^{buy} U_b^{buy}(y_t + u_t; z_t), \qquad (14)$$

subject to

$$\sum_{s=0}^{S} d_{t,s}^{sell} + \sum_{b=0}^{B} d_{t,b}^{buy} \leq 1. \qquad (15)$$

Equation (15) is the constraint that at most one submission is made at t.

B. Optimal Order Submission Strategies

Let $\{d_s^{sell*}(y_t + u_t; z_t),\ d_b^{buy*}(y_t + u_t; z_t)\}$ be the optimal order submission strategy, where the trader's optimal order submission is a function of the trader's valuation and the state vector z_t.

Hollifield et al. (2004) show that the optimal order submission strategy has a monotonicity property. Traders with high private values submit buy orders with high execution probabilities. Traders with low private values submit sell orders with high execution probabilities. Traders with intermediate private values either submit no order, or submit buy or sell limit orders with low execution probabilities.

The optimal order submission strategy is represented in terms of threshold valuations. We can partition the set of valuations into intervals. All traders whose valuations lie within the same interval submit the same order. In order to characterize the intervals, we define a set of threshold valuations that mark the boundaries of the intervals. We determine a trader's optimal order submission simply by identifying which interval the trader's valuation is in.

Define the threshold valuation $\theta_{b,b'}^{buy}(z_t)$ as the valuation of a trader who is indifferent between submitting a buy order at price $p_{t,b}^{buy}$ and a buy order at price $p_{t,b'}^{buy}$, that is,

$$\theta_{b,b'}^{buy}(z_t) = p_{t,b}^{buy} + c_e + \frac{\left(p_{t,b}^{buy} - p_{t,b'}^{buy}\right)\psi_{b'}^{buy}(z_t) + \left(\xi_{b'}^{buy}(z_t) - \xi_b^{buy}(z_t)\right)}{\psi_b^{buy}(z_t) - \psi_{b'}^{buy}(z_t)}. \tag{16}$$

Similarly, the threshold valuation for indifference between a buy order at price $p_{t,b}^{buy}$ and not submitting an order is

$$\theta_{b,no}^{buy}(z_t) = p_{t,b}^{buy} + c_e - \frac{\xi_b^{buy}(z_t) - c_o}{\psi_b^{buy}(z_t)}, \tag{17}$$

the threshold valuation for indifference between a sell order at price $p_{t,s}^{sell}$ and a sell order at price $p_{t,s'}^{sell}$ is

$$\theta_{s,s'}^{sell}(z_t) = p_{t,s}^{sell} - c_e - \frac{\left(p_{t,s'}^{sell} - p_{t,s}^{sell}\right)\psi_{s'}^{sell}(z_t) + \left(\xi_s^{sell}(z_t) - \xi_{s'}^{sell}(z_t)\right)}{\psi_s^{sell}(z_t) - \psi_{s'}^{sell}(z_t)}, \tag{18}$$

the threshold valuation for indifference between a sell order at price $p_{t,s}^{sell}$ and not submitting any order is

$$\theta_{s,no}^{sell}(z_t) = p_{t,s}^{sell} - c_e - \frac{\xi_s^{sell}(z_t) + c_o}{\psi_s^{sell}(z_t)}, \tag{19}$$

and the threshold valuation for indifference between a sell order at price $p_{t,s}^{sell}$ and a buy order at price $p_{t,b}^{buy}$ is

$$\theta_{s,b}(z_t) = \frac{\left(p_{t,b}^{buy}\psi_b^{buy}(z_t) + p_{t,s}^{sell}\psi_s^{sell}(z_t)\right) + c_e\left(\psi_b^{buy}(z_t) - \psi_s^{sell}(z_t)\right) - \left(\xi_b^{buy}(z_t) + \xi_s^{sell}(z_t)\right)}{\psi_s^{sell}(z_t) + \psi_b^{buy}(z_t)}. \tag{20}$$

It may be the case that some order submissions are not optimal for any trader. Let $\mathcal{S}^*(z_t) = \{s_0(z_t), s_1(z_t), s_2(z_t), \ldots, s_S(z_t)\}$ index the set of sell orders that are optimal for some trader at state z_t sorted by their execution probabilities: $1 \geq \psi_{s_0(z_t)}^{sell}(z_t) > \psi_{s_1(z_t)}^{sell}(z_t) > \cdots > \psi_{s_S(z_t)}^{sell}(z_t)$. Define a sell limit order submitted at price $p_{t,s_S(z_t)}^{sell}$ as the marginal sell order. We assume that a sell market order is optimal for traders with some private values and that some sell limit order is optimal for traders with different private values; $\mathcal{S}^*(z_t)$ has at least two elements. Similarly, let $\mathcal{B}^*(z_t)$ index the set of buy orders that are optimal for some trader in state z_t, also sorted by execution probabilities, and define a buy limit order submitted at $p_{t,b_B(z_t)}^{buy}$ as the marginal buy order.

A trader with a valuation lower than $\theta_{b_B(z_t),no}^{buy}(z_t)$, the threshold between a marginal buy order and no order submission, receives a lower expected payoff from submitting any buy order than from submitting no order. A trader with a valuation greater than $\theta_{s_S(z_t),no}^{sell}(z_t)$, the threshold between a marginal sell order and no order submission, receives a lower expected payoff from submitting any sell order than from submitting no order. If $\theta^{sell}_{s_S(z_t),no}(z_t) \le \theta^{buy}_{b_B(z_t),no}(z_t)$, then a trader with a valuation between $\theta^{sell}_{s_S(z_t),no}(z_t)$ and $\theta^{buy}_{b_B(z_t),no}(z_t)$ submits no order. If $\theta^{buy}_{b_B(z_t),no}(z_t) \le \theta^{sell}_{s_S(z_t),no}(z_t)$, then $\theta^{buy}_{b_B(z_t),no}(z_t) \le \theta_{s_S(z_t),b_B(z_t)}(z_t) \le \theta^{sell}_{s_S(z_t),no}(z_t)$, and a trader with any possible valuation submits some order. We therefore define the marginal thresholds for sellers and buyers as

$$\begin{aligned}\theta^{buy}_{marginal}(z_t) &= \max\left(\theta_{s_S(z_t),b_B(z_t)}(z_t),\, \theta^{buy}_{b_B(z_t),no}(z_t)\right),\\ \theta^{sell}_{marginal}(z_t) &= \min\left(\theta_{s_S(z_t),b_B(z_t)}(z_t),\, \theta^{sell}_{s_S(z_t),no}(z_t)\right).\end{aligned} \tag{21}$$

Using the definition of the thresholds, the sell side of the optimal order submission strategy is

$$d^{sell*}_{s}(y_t + u_t; z_t) = 0, \quad \text{for } s \notin \mathcal{S}^*(z_t), \tag{22}$$

$$d^{sell*}_{0}(y_t + u_t; z_t) = \begin{cases} 1, & \text{if } -\infty \le y_t + u_t < \theta^{sell}_{s_0(z_t),s_1(z_t)}(z_t),\\ 0, & \text{otherwise,}\end{cases} \tag{23}$$

$$d^{sell*}_{s_i(z_t)}(y_t + u_t; z_t) = \begin{cases} 1, & \text{if } s_i(z_t) \notin \{0, s_S(z_t)\} \text{ and}\\ & \theta^{sell}_{s_{i-1}(z_t),s_i(z_t)}(z_t) \le y_t + u_t < \theta^{sell}_{s_i(z_t),s_{i+1}(z_t)}(z_t),\\ 0, & \text{otherwise,}\end{cases} \tag{24}$$

$$d^{sell*}_{s_S(z_t)}(y_t + u_t; z_t) = \begin{cases} 1, & \text{if } \theta^{sell}_{s_{S-1}(z_t),s_S(z_t)}(z_t) \le y_t + u_t < \theta^{sell}_{marginal}(z_t),\\ 0, & \text{otherwise,}\end{cases} \tag{25}$$

with the buy side defined similarly.

II. The Gains from Trade

Each trade involves either a sell limit order executing with a buy market order or a sell market order executing with a buy limit order. The gains from a trade are the sum of the traders' realized utilities from the trade. Using equation (8), the gains from trade for a trade at $t+\tau$ between a sell market order submitted at $t+\tau$ by a trader with valuation $u^{sell}_{t+\tau}$ and a buy limit order submitted at t by a trader with valuation u^{buy}_t are

$$\begin{aligned}&\left(p^{sell}_{t+\tau,s} - y_{t+\tau} - u^{sell}_{t+\tau} - c_e - c_o\right) + \left(y_{t+\tau} + u^{buy}_t - p^{buy}_{t,b} - c_e - c_o\right)\\ &\quad = \left(-u^{sell}_{t+\tau} - c_e - c_o\right) + \left(u^{buy}_t - c_e - c_o\right).\end{aligned} \tag{26}$$

The second line follows because $p^{sell}_{t+\tau,s} = p^{buy}_{t,b}$, since the sell market order executes with the buy limit order. The gains from trade for a trade between a sell limit order and a buy market order are computed similarly.

The gains from trade depend neither on the price nor on the common value, since these are simply transferred between the traders. Rather, the gains from trade depend on the difference between the traders' valuations at the time of the trade. The buyer's contribution to the gains from trade is $u_t^{buy} - c_e - c_o$ and the seller's contribution to the gains from trade is $-u_{t+\tau}^{sell} - c_e - c_o$. If a trader submits an order that does not execute, he reduces the gains from trade by c_o.

In the example above, we consider the gains from trade for one possible outcome for the buy limit order submitted by the trader at t. It is also useful to consider the ex ante gains from trade in a given state before the trader's valuation is drawn. Using the distribution of the traders' valuations for the stock and the optimal order submission strategy, we compute expectations over the traders' valuations, the optimal order submissions, and the outcomes of their order submissions. We define the current gains from trade as the expected contribution to the gains from trade in state z_t. Using the execution probabilities, the traders' optimal order submission strategy, and the distribution of the traders' valuations, the expected contribution to the gains from trade for a trader arriving at state z_t is

$$\text{Current gains}\,(z_t) = E\left[\left.\begin{array}{l} \displaystyle\sum_{s=0}^{S} d_s^{sell*}(y_t + u_t; z_t)\big(\psi_s^{sell}(z_t)(-u_t - c_e) - c_o\big) \\ \displaystyle\quad + \sum_{b=0}^{B} d_b^{buy*}(y_t + u_t; z_t)\big(\psi_b^{buy}(z_t)(u_t - c_e) - c_o\big) \end{array}\right| z_t\right]. \tag{27}$$

The current common value, the limit and market order prices, and the state vector z_t enter through their effects on the traders' optimal order submission strategy.

The maximum gains from trade are determined by finding the post-trade allocation of the stock among the traders that results in the maximum expected gains from trade. The maximum gains from trade may not be achievable by any mechanism because of the inherent frictions caused by traders arriving sequentially with trading opportunities that last for a finite period of time. Incentive compatibility issues will typically further reduce the gains from trade attainable in any feasible mechanism. The maximum gains from trade is easy to compute and provides a useful upper bound on the gains from trade in any feasible mechanism.

To describe a stock allocation, define the sell indicator function as

$$I^{sell}(u_t; x_t) = \begin{cases} 1, & \text{if a trader with private value } u_t \text{ sells the stock in state } x_t, \\ 0, & \text{otherwise,} \end{cases} \tag{28}$$

and define the buy indicator function $I^{buy}(u_t; x_t)$ similarly.

Using the sell and buy indicators, the allocation that maximizes the gains from trade solves

$$\max_{\{I^{sell}(u_t;x_t), I^{buy}(u_t;x_t)\}} E\big[I^{sell}(u_t; x_t)(-u_t - c_e - c_o) + I^{buy}(u_t; x_t)(u_t - c_e - c_o)\,\big|\,x_t\big], \tag{29}$$

subject to

$$I^{sell}(u_t;x_t) + I^{buy}(u_t;x_t) \leq 1, \text{ for all } u_t, \tag{30}$$

$$E\big[I^{sell}(u_t;x_t)\,\big|\,x_t\big] = E\big[I^{buy}(u_t;x_t)\,\big|\,x_t\big]. \tag{31}$$

Equation (30) is the constraint that each trader has a single opportunity to trade and equation (31) is the market clearing condition.

The optimal allocation and the maximum gains from trade with a symmetric distribution for the private values with median zero are reported in the following lemma.

LEMMA 1: *Suppose that the private values are drawn from the continuous, symmetric distribution $G(\cdot\,|\,x_t)$ with median zero. The allocation that solves (29) subject to (30) and (31) is*

$$I^{sell*}(u_t;x_t) = \begin{cases} 1, & \text{for } u_t \leq -c_e - c_o \\ 0, & \text{else}, \end{cases} \quad , \quad I^{buy*}(u_t;x_t) = \begin{cases} 1, & \text{for } u_t \geq c_e + c_o \\ 0, & \text{else}. \end{cases} \tag{32}$$

The maximum gains from trade are

$$\begin{aligned} &\text{Maximum gains } (x_t) \\ &\quad = E\big[I^{sell*}(u_t;x_t)(-u_t - c_e - c_o) + I^{buy*}(u_t;x_t)(u_t - c_e - c_o)\,\big|\,x_t\big]. \end{aligned} \tag{33}$$

The proof is in Appendix A.

By construction, the current gains from trade in the limit order market are less than or equal to the maximum gains from trade. The current gains may be less than the maximum gains because limit orders face execution risk and the traders' private incentives may lead them to make order submissions that are different from the ones that would lead to the social optimum. We decompose the differences between the maximum and current gains into four sources: no execution, no submission, wrong direction, and extramarginal submission.

No execution is the expected loss from traders who buy or sell the stock in the optimal allocation but whose buy or sell orders do not execute in the limit-order market, that is,

$$\begin{aligned} &\text{No execution } (z_t) \\ &= E\left[\left.\begin{aligned} &I^{sell*}(u_t;x_t)\sum_{s=0}^{S} d_s^{sell*}(y_t+u_t;z_t)\big(1-\psi_s^{sell}(z_t)\big)(-u_t-c_e) \\ &\quad + I^{buy*}(u_t;x_t)\sum_{b=0}^{B} d_b^{buy*}(y_t+u_t;z_t)\big(1-\psi_b^{buy}(z_t)\big)(u_t-c_e) \end{aligned}\right|\, z_t\right]. \end{aligned} \tag{34}$$

Losses from no execution arise because it is sometimes individually optimal for traders with valuations that differ from the common value by more than $c_e + c_o$ to submit limit orders that may fail to execute.

No submission is the expected loss from traders who buy or sell the stock in the optimal allocation but do not submit an order in the limit-order market, that is,

$$\text{No submission}(z_t) = E\left[\left.\begin{array}{l}\left(1-\sum_{s=0}^{S} d_s^{sell*}(y_t+u_t;z_t)-\sum_{b=0}^{B} d_b^{buy*}(y_t+u_t;z_t)\right)\\ \times\left(I^{sell*}(u_t;x_t)(-u_t-c_e-c_o)+I^{buy*}(u_t;x_t)(u_t-c_e-c_o)\right)\end{array}\right| z_t\right]. \quad (35)$$

Losses from no submission arise because it is sometimes individually optimal for traders with valuations that differ from the common value by more than $c_e + c_o$ to not submit any order.

Although a trader's order submission is individually optimal by construction, it need not lead to a positive contribution to the gains from trade. For example, suppose a sell market order and a buy limit order transact at price $p_{t,0}^{sell}$. If the seller's valuation $u_t^{sell} > 0$, the seller makes a negative contribution to the gains from trade because $-u_t^{sell} - c_e - c_o < 0$. Nevertheless, the trade can be individually optimal for the seller if $p_{t,0}^{sell} - y_t > c_e + c_o$. In the example, a trader with a positive private value—and hence, no particular need to sell—may end up selling the stock because the limit-order book provides an opportunity to sell at a sufficiently high price.

The seller transacts with a buy limit order submitted by a previous trader with a high valuation; thus the common value at the time of the buy limit order submission or the previous trader's private value or both are high. Depending on the seller's private value, his trade contributes either to the wrong direction losses or to the extramarginal submissions that we define below.

Wrong direction is the expected loss from traders who buy or sell the stock in the optimal allocation but submit an order to trade in the wrong direction in the limit-order market:

$$\text{Wrong direction }(z_t) = E\left[\left.\begin{array}{l} I^{sell*}(u_t;x_t)\sum_{b=0}^{B} d_b^{buy*}(y_t+u_t;z_t)\left(-u_t-c_e+\psi_b^{buy}(z_t)(-u_t+c_e)\right)\\ +I^{buy*}(u_t;x_t)\sum_{b=0}^{B} d_b^{sell*}(y_t+u_t;z_t)\left(u_t-c_e+\psi_s^{sell}(z_t)(u_t+c_e)\right)\end{array}\right| z_t\right]. \quad (36)$$

Extramarginal submission is the expected loss from traders who do not trade in the optimal allocation but submit buy or sell orders in the limit-order market:

Extramarginal submission (z_t)

$$= E\left[\left.\begin{array}{l}\left(1 - I^{sell*}(u_t;x_t) - I^{buy*}(u_t;x_t)\right)\\ \times\left(\begin{array}{l}\displaystyle\sum_{s=0}^{S} d_s^{sell*}(y_t+u_t;z_t)\left(\psi_s^{sell}(z_t)(u_t+c_e)+c_o\right)\\ \displaystyle+\sum_{b=0}^{B} d_b^{buy*}(y_t+u_t;z_t)\left(\psi_b^{buy}(z_t)(-u_t+c_e)+c_o\right)\end{array}\right)\end{array}\right| z_t\right]. \tag{37}$$

Another useful comparison to the gains from trade in the limit order market is the gains from trade that would obtain if bid and ask quotes were made by a profit-maximizing monopolist with a private value of zero and no inventory carrying costs. The monopoly gains from trade are determined by finding the monopolist's profit-maximizing bid and ask quotes and computing the gains from trade at the resulting allocation.

To describe the monopolist's problem, let b_t^m denote the bid quote and a_t^m the ask quote that the monopolist dealer quotes at time t. To describe a traders' decision, define the sell indicator function

$$I^{m,sell}\left(b_t^m;u_t;x_t\right) = \begin{cases}1, & \text{if a trader with private value } u_t \text{sells the stock in state } x_t\\ 0, & \text{otherwise,}\end{cases} \tag{38}$$

and define the buy indicator function $I^{m,buy}(a_t^m;u_t;x_t)$ similarly.

Using the sell and buy indicators, the monopolist's profit-maximizing quotes solve

$$\max_{\{b_t^m,a_t^m\}} E\left[I^{m,sell}\left(b_t^m;u_t;x_t\right)\left(y_t - b_t^m\right) + I^{m,buy}\left(a_t^m;u_t;x_t\right)\left(a_t^m - y_t\right)\right], \tag{39}$$

subject to

$$E\left[I^{m,sell}\left(b_t^m;u_t;x_t\right) \,\middle|\, x_t\right] = E\left[I^{m,buy}\left(a_t^m;u_t;x_t\right) \,\middle|\, x_t\right]. \tag{40}$$

LEMMA 2: *Suppose the private values are drawn from the continuous, symmetric distribution $G(\cdot\,|\,x_t)$ with median zero. The monopolist's optimal quotes a_t^{m*} and b_t^{m*} solve*

$$b_t^{m*} = y_t - \frac{G\left(b_t^{m*} - c_e - c_o - y_t \,\middle|\, x_t\right)}{g\left(b_t^{m*} - c_e - c_o - y_t \,\middle|\, x_t\right)} \tag{41}$$

and

$$a_t^{m*} = y_t + \frac{1 - G\left(a_t^{m*} + c_e + c_o - y_t \,\middle|\, x_t\right)}{g\left(a_t^{m*} + c_e + c_o - y_t \,\middle|\, x_t\right)}. \tag{42}$$

The traders' decisions given b_t^{m*} and *a_t^{m*} are*

$$I^{m,sell}\left(b_t^{m*};u_t;x_t\right) = \begin{cases}1, & \text{for } u \le b_t^{m*} - c_e - c_o - y_t\\ 0, & \text{otherwise}\end{cases}, \tag{43}$$

and

$$I^{m,buy}\left(a_t^{m*};u_t;x_t\right)=\begin{cases}1, & \text{for } u\geq a_t^{m*}+c_e+c_o-y_t\\ 0, & \text{otherwise}\end{cases}, \tag{44}$$

assuming that the second-order conditions reported in equations (A14) and (A15) are satisfied at the solutions. The monopoly gains from trade are

$$\text{Monopoly gains }(x_t)$$
$$=E\left[I^{m,sell}\left(b_t^{m*};u_t;x_t\right)(-u_t-c_e-c_o)+I^{m,buy}\left(a_t^{m*};u_t;x_t\right)(u_t-c_e-c_o)\,\middle|\,x_t\right]. \tag{45}$$

The proof is in Appendix A.

From equations (41) and (42), $b_t^{m*}<y_t$ and $a_t^{m*}>y_t$. Comparing the monopolist allocation with the allocation that maximizes the gains from trade in equation (32), the monopolist sets bid and ask quotes that results in less trading than is efficient.

III. Empirical Results

We use a two-step method to estimate the parameters of the model. In the first step, we use the execution and cancellation histories of the order submissions to estimate the execution probabilities and picking-off risks. In the second step, we estimate the private value distributions, traders' arrival rates, and costs by maximizing the conditional log-likelihood function for limit and market order arrival times. We use the estimated parameters to form estimates of the current, maximum, and monopoly gains from trade.

A. Description of the VSE and Our Sample

Our sample comes from the audit tapes of the VSE. The limit order trading system used by the VSE is very similar to the systems used by the London Stock Exchange, Euronext, the Stockholm Stock Exchange, the Toronto Stock Exchange, the Archipelago Exchange and many other markets. In 1999, after the end of our sample, the VSE was involved in an amalgamation of Canadian equity trading and became part of the Canadian Venture Exchange, which in turn was recently renamed the TSX Venture Exchange. The TSX Venture Exchange is similar in structure, rules, and activity to the VSE.

Forty-five exchange member firms act as brokers, submitting orders for outside traders, and as dealers, submitting orders on their own account. There are no designated market makers. The market is open from 6:30 a.m. to 1:30 p.m. Pacific time. Limit orders in the order book are matched with incoming market orders to produce trades, giving priority to limit orders according to the order price and then the time of submission. Order prices must be multiples of a tick size. The tick size varies between one cent for prices below \$3.00, five cents for prices between \$3.00 and \$4.99, and twelve-and-a-half cents for prices at \$5.00

Table I
Order Submissions

The sample period is May 1, 1990, to November 30, 1993.

Stock Ticker	Barkhor Resources BHO	Eurus Resources ERR	War Eagle Mining Company WEM
Number of order submissions	55,444	56,599	47,578
Percent of submissions			
Sell limit orders	31.7	31.5	31.3
Sell market orders	21.6	23.7	19.9
Buy limit orders	28.5	27.9	32.0
Buy market orders	18.2	16.9	16.8
Time between order submissions in seconds			
Average	247.0	297.9	396.6
Standard deviation	842.4	983.7	1089.4

and above. Orders sizes must be multiples of a fixed size which varies between 100 and 1,000 shares.

Member firms can also submit hidden limit orders, where a fraction of the order size is not visible on the limit order book. The hidden fraction of the order retains its price priority, but loses its time priority. In our sample, few hidden orders are submitted, thus the assumption of no hidden limit orders in our model is a reasonable approximation for our sample.

Our sample contains a record for every trade, cancellation, or change in the status of an order, and the limit-order book at the open of each day. Each record includes the time of the original order submission, but not the member firms' identification codes nor whether or not a member firm submitted an order as a broker or dealer. Combining the records with the limit-order book at the open of each day we reconstruct order histories for each order submission, including the initial order submission and every future order execution or cancellation, and the corresponding order books. Less than 1% of the orders are associated with inconsistencies between the inferred order histories and the trading rules. We drop such orders from our sample.

Our sample goes from May 1990 to November 1993 for three stocks in the mining industry. Table I reports the name and ticker symbol of the three stocks. The table reports the total number of order submissions, the percentage of buy and sell market and limit orders submitted in our sample, and the average and standard deviation of the time between order submissions.

B. Construction of the Variables

We use a centered moving average of the mid-quotes over a 20-minute window to proxy for the stock's unobserved common value. Our proxy is reasonable because most of the time the best quotes should straddle the common value.

We use a centered moving average to reduce the impact of mechanical shifts in the mid-quote caused by individual order submissions or cancellations.

Table II reports our choice for the state vector $z_t = (x_t, w_t)$. The table reports the names of the variables, a brief description of them, and their sample means and standard deviations. In the theoretical model, the exogenous state variables, x_t, predict the trader arrival rates, the distribution of innovations to the common value, and the conditional distributions of the traders' private values. Our choice of exogenous state variables is reported in the top panel of Table II. We observe the exogenous state variables at a daily frequency.

We select exogenous state variables that are likely to be correlated with the traders' desire to change their portfolios and with innovations in the stocks' common value. In particular, the exogenous state variables that we use are Toronto Stock Exchange (TSX) market index volatility, TSX mining volatility, interest rate volatility, exchange rate volatility, and stock volatility. Traders may be more likely to want to change their portfolio as a result of changes in the market index, interest rates, or the stock price. For example, a change in the market index may lead more traders to wish to change their portfolios and may also change the traders' willingness to pay more to buy immediately or receive less to sell immediately. Such effects are captured by changes in the trader arrival rate and the distribution of private values.

The bottom panel in Table II reports our choice of endogenous state variables, w_t. Ideally, w_t would include the entire limit-order book and any other variables known at t that are useful for predicting the outcomes of order submissions at t. However, because our sample size is relatively small we use a smaller number of variables in w_t.

The bid–ask spread and measures of depth close to the quotes and away from the quotes directly measure the state of the limit order book. Close depth is the number of shares outstanding at the current best quotes and far depth is the cumulative number of shares outstanding up to and including the marginal limit order. The depth measures on the same side of the book often contain very similar information. We therefore in our first-step estimation include only one depth measure for each side of the market. We use the depth in front of the order as well as the close depth on the other side of the book to predict the execution probabilities and picking-off risks. Execution probabilities depend on book variables through the length of the order queues at different prices. Execution probabilities also depend on book variables indirectly through the book's effect on the current and future order submissions and cancellations.

Past order submission activity is useful to predict the latent execution and cancellation times for new order submissions because the average age of existing limit orders in the book can influence the probability that the existing limit orders are canceled. We use the number of recent trades and lagged durations to measure past order submission activity. Holding everything else equal, larger orders are likely to have lower execution probabilities and to face higher picking-off risk. The distance between the current mid-quote and our proxy for the common value is included because holding everything else equal, a buy

Table II

The State Vector, z_t

The table describes the construction of the state vector $z_t = (x_t, w_t)$, with x_t, the exogenous state variables, and w_t, the endogenous state variables. We also report the means and standard deviations of all the variables for the three stocks in our sample: Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

		BHO		ERR		WEM	
Name	Description	Mean	*SD*	Mean	*SD*	Mean	*SD*
	Exogenous State Variables: x_t						
TSX market volatility	Absolute value of the lagged close-to-close return on the TSX market index	0.39	0.35	0.39	0.35	0.39	0.34
TSX mining volatility	Absolute value of the lagged close-to-close return on the TSX mining index	1.29	1.19	1.30	1.19	1.29	1.19
Interest rate volatility	Absolute value of the lagged change in the overnight interest rate	0.16	0.21	0.16	0.21	0.16	0.21
Exchange rate volatility	Absolute value of the lagged change in the Canadian/U.S. exchange rate	0.20	0.19	0.20	0.18	0.20	0.19
Stock volatility	Absolute value of the lagged open-to-open stock return	4.28	8.89	4.16	6.11	3.59	5.04
	Endogenous State Variables: w_t						
Spread	Bid–ask spread	0.02	0.02	0.05	0.06	0.04	0.03
Close ask depth	Logarithm of ask depth at best ask	1.74	0.95	0.94	0.79	1.02	0.75
Far ask depth	Logarithm of cumulative ask depth up to the marginal sell order	4.04	0.83	2.60	0.81	2.83	0.69
Close bid depth	Logarithm of bid depth at best bid	1.75	1.00	1.04	0.85	1.08	0.83
Far bid depth	Logarithm of cumulative bid depth down to the marginal buy order	4.06	0.78	2.82	0.85	3.14	0.77
Order quantity	Logarithm of the number of shares in the order	1.49	0.83	0.72	0.90	0.84	0.83
Recent trades	Number of trades in the last 10 minutes	8.61	10.88	5.06	6.75	4.02	5.68
Lagged duration	Sum of last 10 durations of order book changes, divided by 1,000	3.10	5.30	3.16	4.96	3.93	5.20
Mid-quote volatility	Volatility of the mid-quote over the last 10 minutes	1.71	0.07	1.71	0.09	1.70	0.05
Distance to mid-quote	Moving average mid-quote minus the mid-quote, measured as a percentage	0.00	0.02	0.00	0.04	0.00	0.02
	Time Dummies						
First hour	Dummy variable for 6:30–7:30						
Second hour	Dummy variable for 7:30–8:30						
Third hour	Dummy variable for 8:30–9:30						
Fourth hour	Dummy variable for 9:30–10:30						
Fifth hour	Dummy variable for 10:30–11:30						
Sixth hour	Dummy variable for 11:30–12:30						

order at two ticks below the common value is less likely to be executed than a buy order one tick below the common value.

We include six hourly dummy variables to capture any deterministic time-of-day patterns in execution probabilities and picking-off risks. Deterministic time-of-day patterns may arise because of deadline effects associated with market closure. For example, some traders cancel unexecuted limit orders at the close of the market. Such behavior introduces time-of-day patterns in the timing of cancellations, as orders submitted early are less likely to remain outstanding at the time of the market close than orders submitted later.

We assume that some traders always find it optimal to submit one-tick buy and sell limit orders. In the theoretical model, the marginal sell limit order is defined as the highest-priced sell limit order that any trader would optimally submit at z_t. The marginal buy limit order is defined similarly. Empirically, we set the marginal prices depending on the level of the common value. For each decile of the common value, we define a cutoff sell price as the price such that at least 95% of the sell order prices are below that cutoff sell price. The marginal sell order is defined as the lowest-priced sell order above the cutoff price. The marginal buy order is defined similarly.

We purposely drop 5% of the order submissions to avoid a situation in which the orders submitted at extreme prices—which may represent order entry mistakes—would determine the marginal order prices. By ignoring limit orders submitted outside the price range defined by the marginal prices, we ignore the expected payoffs received by traders whose private values would lead them to submit limit orders outside that price range. Omitting some payoffs may lead to a downward bias in our estimates of the current gains from trade, but should not affect our estimates of the maximum gains from trade and the monopoly gains from trade.

We need to estimate the execution hazard rate, the cancellation hazard rate, and the expected change in the common value conditional on execution for the marginal limit orders. By construction, the marginal limit orders are a small fraction of the orders submitted and tend to have a low execution probability. In estimating the execution hazard rate, the cancellation hazard rate, and the expected change in the common value conditional on execution for the marginal limit orders, we combine orders submitted at the marginal price with orders submitted up to two ticks away from the marginal price. By grouping orders, we ensure that we have enough executed orders to obtain estimates of the hazard rates and the expected change in the common value conditional on execution for the marginal limit orders.

In the theoretical model, order quantity is normalized to one unit: All orders are either fully executed or canceled. In our sample, however, different order submissions have different order quantities, that is, partial executions occur. Empirically, we handle partial executions by assuming an order was fully executed if at least 50% of the order size is executed and fully cancelled otherwise. Table III reports the average percentage of the submitted limit order quantity that is executed within 48 hours, conditional on that percentage being at least 50% or less than 50%. Less than 1% of the order executions occur more than

Table III

Order Executions and Cancellations

The top panel reports for all limit orders the average percentage of the submitted order quantity that is executed within 48 hours of order submission, conditional on that percentage being at least 50% or less than 50%. Standard errors are reported in parentheses. The second panel reports, for all limit orders that eventually execute, the percent that execute within five different time intervals. For orders that have several partial executions, the time of execution is weighted by the quantity executed to determine an average execution time. The third panel reports, for all orders that are eventually cancelled, the percent that cancel within five time intervals relative to the time of the order submission, the percent that cancel within five time intervals before the market close on the day of order submission, and the percent that cancel the next day or later. The three sample stocks are Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

	BHO	ERR	WEM
Conditional Order Execution Percentage			
At least 50% executes	97.34	96.97	96.58
	(0.07)	(0.07)	(0.09)
Less than 50% executes	1.40	1.43	1.21
	(0.04)	(0.04)	(0.04)
Distribution of Time to Execution			
Percent of orders executed			
Less than 5 minutes after order submission	46.67	42.77	37.29
5 to 15 minutes after order submission	15.92	17.97	17.24
15 minutes to 1 hour after order submission	19.59	21.92	22.60
1 to 3 hours after order submission	10.73	11.46	14.07
More than 3 hours after order submission	7.09	5.88	8.80
Distribution of Time to Cancellation			
Percent of orders cancelled			
Less than 5 minutes after order submission	31.79	27.57	23.49
5 to 15 minutes after order submission	13.08	13.81	11.78
15 minutes to 1 hour after order submission	16.42	18.17	16.64
1 to 3 hours after order submission	14.30	15.45	17.70
More than 3 hours after order submission	24.41	25.00	30.39
Percent of orders cancelled			
Less than 5 minutes before market close	18.77	21.16	26.32
5 to 15 minutes before market close	2.84	2.51	2.80
15 minutes to 1 hour before market close	9.15	8.68	9.46
1 to 3 hours before market close	20.80	20.31	20.35
More than 3 hours before market close	40.78	41.12	33.05
After the first day market close	7.66	6.22	8.02

48 hours from the time of the order submission. The average execution percentage is close to 100% conditional on the execution percentage being 50% or more, and the average execution percentage is close to 0% conditional on the execution percentage being less than 50%. Our assumption with respect to partial execution is therefore a reasonable approximation of our sample.

In the theoretical model, both execution and cancellation times are random. The assumption of random execution and cancellation times is also justified in our sample. The second panel of Table III, which reports the distribution of the time to execution for all limit orders in our sample shows that time to execution is indeed random. Similarly, the third panel of Table III shows that the time to cancellation is also random; orders are not canceled a fixed number of minutes after submission nor are they all canceled at the end of the trading day.

C. Estimates of the Execution Probabilities and Picking-Off Risks

We assume that the traders' beliefs about the execution probabilities and picking-off risks are consistent with the empirical execution and cancellation histories.

The execution probabilities are determined by the distributions of the latent times to cancellation and execution in equations (5) and (6). The picking-off risks are determined by the execution probabilities and the expected change in the common value conditional on the order executing. We use our sample to estimate both the distribution of the latent times and the expected change in the common value conditional on the order executing. We then use the resulting estimates to compute estimates of execution probabilities and picking-off risks. In turn, the estimates of the execution probabilities and picking-off risks are used to characterize the traders' expectations and the optimal order submission strategy.

Our formulation of independent latent execution and cancellation times is a competing risks model. Appendix B provides a brief description of the competing risks model. We use the distribution of the latent execution and cancellation times to compute the execution probabilities. Our approach extends Cho and Nelling (2000), who compute execution probabilities for limit orders based on the parameter estimates for the distribution of the time to execution, assuming that all orders are canceled at the end of the day.

We parameterize the conditional distributions of the cancellation times as a Weibull,

$$F_{cancel}\left(\tau \,\middle|\, z_t, d_{t,b}^{buy} = 1\right) = 1 - \exp\left(-\exp\left(z_t' \gamma_b^{buy}\right) \tau^{\alpha_b^{buy}}\right), \tag{46}$$

where z_t is the state vector. The hazard rate is defined as the probability that the cancellation time occurs between $t + \tau$ and $t + \tau + d\tau$, conditional on the cancellation time being greater than $t + \tau$. For the Weibull model, the hazard rate is

$$\Pr\left(\tau_{cancel} \in [\tau, \tau + d\tau) \,\middle|\, \tau_{cancel} \geq \tau, z_t, d_{t,b}^{buy} = 1\right) = \exp\left(z_t' \gamma_b^{buy}\right) \alpha_b^{buy} \tau^{\alpha_b^{buy} - 1} \, d\tau. \tag{47}$$

The parameter vector γ_b^{buy} measures the effect of the state vector on the hazard rate. If a variable has a positive parameter, then an increase in that variable increases the hazard rate. The parameter α_b^{buy} is the Weibull shape parameter. If $\alpha_b^{buy} = 1$, the hazard rate does not depend on τ. If $\alpha_b^{buy} < 1$, the hazard rate is decreasing in τ. If $\alpha_b^{buy} > 1$, the hazard rate is increasing in τ.

Table IV reports the results for the cancellation time distributions. The models are estimated for one-tick and marginal limit orders and are estimated by maximum likelihood. We treat orders that last longer than 2 days as censored observations.

The parameter estimates for the Weibull shape parameters are all less than 1, with an average value of 0.56. The cancellation hazard rates are decreasing in the time that the order is in the book. The age of an order predicts the conditional probability of cancellation. Past activity is correlated with the age of unfilled orders in the book. Past activity therefore can predict the cancellation rates of existing orders in the book, consistent with the assumption in the theoretical model.

We also parameterize the conditional distributions of the time to execution as a Weibull

$$F_{execute}\big(\tau \,\big|\, z_t, d_{t,b}^{buy} = 1\big) = 1 - \exp\big(-\exp\big(z_t' \kappa_b^{buy}\big)\tau^{\beta_b^{buy}}\big). \tag{48}$$

The conditional distributions of the time to execution depend on the trader arrival rates, the order cancellation distributions, and future traders' order submissions. A disadvantage of the parametric model is that it imposes auxiliary restrictions on the conditional distributions. An alternative, which does not impose such auxiliary assumptions, is to use nonparametric methods as in Hollifield et al. (2004). An advantage of the parametric model is that we can use a larger state vector than with a nonparametric method. We therefore choose the parametric model because it allows us to approximate the large information set available to the traders.

Table V reports the results for the execution time distributions. The models are estimated for one-tick and marginal limit orders and are estimated by maximum likelihood. We treat the orders that last longer than 2 days as censored observations.

The parameter estimates for the Weibull shape parameters are all less than 1, with an average value of 0.65. The execution hazard rates are decreasing in the time that the order is in the book. The Weibull shape parameter is lower for the cancellation times than for the execution times; the probability that a limit order is canceled rather than executed decreases with the time the order is in the book.

Tables IV and V also report chi-squared tests for the null hypotheses that the conditional distributions of the execution and cancellation times do not depend on the state vector z_t. We reject the null hypothesis for all order submissions and stocks.

We use the parameter estimates from the conditional distributions of the execution times and cancellation times to forecast the execution probabilities for buy and sell one-tick and marginal limit orders at every order submission. We compute the probability that the order executes within 2 days: $T = 2$ days. Appendix C reports details of the computations of the execution probabilities.

For Barkhor Resources (BHO), the average execution probability for marginal sell limit orders is approximately 16%, for one-tick sell limit orders is 61%, for

Table IV

Weibull Model for Cancellation Times

Parameter estimates with asymptotic standard errors in parentheses for a Weibull model for the cancellation times. The χ^2 test is for the null that the state vector z_t does not affect the conditional distribution with p-values in parentheses. The three sample stocks are Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

	BHO				ERR				WEM			
	Sell Orders		Buy Orders		Sell Orders		Buy Orders		Sell Orders		Buy Orders	
Variable	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick
Shape	0.63	0.51	0.62	0.52	0.62	0.50	0.59	0.54	0.62	0.49	0.65	0.47
	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)
Constant	−5.08	−1.72	−6.03	−6.93	−6.64	−5.60	−4.23	−2.76	−4.51	1.11	−3.55	−9.61
	(0.89)	(0.85)	(0.63)	(0.73)	(0.98)	(1.21)	(0.66)	(0.81)	(0.91)	(1.76)	(1.09)	(1.96)
TSX market	0.01	0.02	−0.01	−0.00	0.05	0.00	0.04	0.05	−0.01	0.01	0.01	0.03
volatility	(0.03)	(0.03)	(0.03)	(0.03)	(0.02)	(0.03)	(0.03)	(0.04)	(0.03)	(0.04)	(0.03)	(0.04)
TSX mining	0.06	0.03	0.01	0.00	−0.00	0.01	−0.05	−0.12	0.01	0.07	0.00	−0.01
volatility	(0.03)	(0.02)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.04)	(0.03)	(0.04)	(0.03)	(0.04)
Interest rate	0.00	0.01	0.02	−0.03	0.05	0.05	−0.00	0.05	0.03	−0.00	−0.03	0.00
volatility	(0.03)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.03)	(0.05)	(0.03)	(0.04)	(0.03)	(0.04)
Exchange rate	−0.02	0.02	−0.06	−0.05	−0.03	−0.04	−0.02	−0.08	−0.07	−0.01	−0.06	−0.03
volatility	(0.03)	(0.02)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.04)	(0.03)	(0.04)	(0.03)	(0.05)
Stock	0.00	0.07	0.07	0.09	−0.00	−0.06	−0.03	0.01	0.01	0.00	0.04	0.07
volatility	(0.03)	(0.02)	(0.03)	(0.03)	(0.02)	(0.04)	(0.03)	(0.03)	(0.03)	(0.04)	(0.03)	(0.04)
Spread	−0.54	−9.19	3.46	2.20	2.21	−5.59	2.71	0.21	3.54	3.64	2.11	−7.16
	(1.48)	(4.21)	(1.69)	(4.33)	(0.71)	(2.22)	(0.63)	(2.70)	(1.08)	(3.02)	(0.83)	(3.95)
Close ask	—	−0.01	0.01	0.05	—	0.00	−0.05	−0.05	—	−0.08	−0.02	0.04
depth	—	(0.03)	(0.04)	(0.03)	—	(0.04)	(0.03)	(0.04)	—	(0.05)	(0.04)	(0.05)
Far ask	−0.15	—	—	—	−0.06	—	—	—	−0.15	—	—	—
depth	(0.04)	—	—	—	(0.03)	—	—	—	(0.04)	—	—	—
Close bid	−0.01	0.00	—	−0.01	−0.01	0.02	—	−0.02	−0.05	0.08	—	−0.08
depth	(0.03)	(0.02)	—	(0.03)	(0.03)	(0.03)	—	(0.04)	(0.04)	(0.04)	—	(0.04)

(continued)

Table IV—*Continued*

	BHO				ERR				WEM			
	Sell Orders		Buy Orders		Sell Orders		Buy Orders		Sell Orders		Buy Orders	
Variable	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick
Far bid	—	—	−0.08	—	—	—	−0.07	—	—	—	−0.09	—
depth	—	—	(0.04)	—	—	—	(0.03)	—	—	—	(0.04)	—
Order	0.04	−0.05	0.03	−0.12	0.14	−0.03	0.01	−0.02	0.06	−0.06	−0.05	−0.10
quantity	(0.04)	(0.03)	(0.04)	(0.03)	(0.03)	(0.03)	(0.03)	(0.04)	(0.04)	(0.05)	(0.03)	(0.04)
Recent	0.01	0.03	0.02	0.03	0.02	0.01	0.03	0.04	0.04	0.04	0.03	0.04
trades	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Lagged	−0.11	−0.02	−0.08	−0.05	−0.06	−0.01	−0.06	−0.04	−0.05	−0.03	−0.05	−0.03
duration	(0.01)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Mid-quote	0.10	−1.09	0.71	1.99	1.04	1.53	−0.08	−0.24	−0.17	−2.74	−0.95	3.84
volatility	(0.49)	(0.50)	(0.36)	(0.43)	(0.56)	(0.71)	(0.38)	(0.46)	(0.52)	(1.03)	(0.64)	(1.15)
Distance to	0.22	21.25	−2.83	−11.92	9.41	25.91	−9.10	−31.00	7.16	31.78	−5.01	−12.41
mid-quote	(2.08)	(2.19)	(1.73)	(2.12)	(2.93)	(5.05)	(2.47)	(6.37)	(2.54)	(3.91)	(2.73)	(5.57)
First hour	−0.67	−1.12	−1.04	−1.01	−1.07	−0.82	−1.09	−0.72	−1.02	−0.96	−1.03	−0.62
	(0.11)	(0.09)	(0.13)	(0.09)	(0.10)	(0.11)	(0.10)	(0.13)	(0.11)	(0.14)	(0.11)	(0.16)
Second hour	−0.66	−0.96	−0.87	−1.11	−0.97	−1.03	−1.02	−0.81	−1.01	−0.80	−0.69	−1.07
	(0.12)	(0.08)	(0.13)	(0.09)	(0.10)	(0.11)	(0.10)	(0.13)	(0.11)	(0.13)	(0.10)	(0.15)
Third hour	−0.37	−0.90	−0.76	−1.00	−0.91	−0.97	−0.94	−1.09	−0.73	−0.87	−0.71	−0.68
	(0.12)	(0.08)	(0.14)	(0.09)	(0.10)	(0.12)	(0.10)	(0.14)	(0.11)	(0.13)	(0.11)	(0.12)
Fourth hour	−0.35	−0.86	−0.77	−0.77	−0.61	−0.82	−0.77	−1.02	−0.74	−0.73	−0.48	−0.55
	(0.12)	(0.09)	(0.14)	(0.08)	(0.10)	(0.11)	(0.11)	(0.13)	(0.11)	(0.13)	(0.11)	(0.12)
Fifth hour	−0.05	−0.64	−0.43	−0.89	−0.77	−0.63	−0.64	−0.77	−0.50	−0.56	−0.33	−0.85
	(0.12)	(0.08)	(0.14)	(0.08)	(0.10)	(0.11)	(0.11)	(0.13)	(0.12)	(0.13)	(0.12)	(0.13)
Sixth hour	0.13	−0.46	−0.25	−0.65	−0.38	−0.49	−0.37	−0.77	−0.33	−0.41	−0.12	−0.49
	(0.12)	(0.08)	(0.15)	(0.09)	(0.10)	(0.11)	(0.11)	(0.13)	(0.11)	(0.12)	(0.11)	(0.12)
$\chi^2(19)$	290.01	576.14	247.37	648.10	258.47	211.25	304.15	218.13	273.52	188.44	273.48	201.66
p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
No. of Obs.	1,748	4,498	1,353	4,105	2,458	2,368	2,138	1,732	1,679	1,793	1,792	1,689

Table V

Weibull Model for Execution Times

Parameter estimates, with asymptotic standard errors in parentheses, for a Weibull model for the execution times. The χ^2 test is for the null that the state vector z_t does not affect the conditional distribution, with p-values in parentheses. The three sample stocks are Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

	BHO				ERR				WEM			
	Sell Orders		Buy Orders		Sell Orders		Buy Orders		Sell Orders		Buy Orders	
Variable	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick
Shape	0.66	0.58	0.70	0.61	0.65	0.54	0.82	0.64	0.64	0.58	0.79	0.59
	(0.03)	(0.01)	(0.04)	(0.01)	(0.02)	(0.01)	(0.03)	(0.02)	(0.03)	(0.01)	(0.04)	(0.01)
Constant	−15.69	−10.42	−4.31	2.04	−12.97	−10.35	−3.86	3.93	−5.93	−26.41	1.22	12.91
	(1.51)	(0.42)	(1.85)	(0.47)	(0.89)	(0.58)	(1.23)	(1.86)	(1.60)	(1.94)	(2.01)	(2.01)
TSX market	0.07	−0.00	0.05	0.02	0.18	0.03	0.00	−0.03	−0.04	0.04	0.07	0.00
volatility	(0.05)	(0.02)	(0.08)	(0.02)	(0.04)	(0.03)	(0.05)	(0.03)	(0.06)	(0.04)	(0.06)	(0.04)
TSX mining	−0.03	−0.06	0.01	−0.04	−0.16	0.01	0.01	−0.02	−0.26	−0.04	0.10	0.05
volatility	(0.07)	(0.02)	(0.08)	(0.02)	(0.05)	(0.03)	(0.05)	(0.03)	(0.07)	(0.04)	(0.06)	(0.04)
Interest rate	−0.03	−0.02	0.06	0.02	−0.09	−0.08	0.08	−0.02	0.02	0.01	0.06	0.09
volatility	(0.07)	(0.02)	(0.09)	(0.02)	(0.05)	(0.04)	(0.04)	(0.05)	(0.05)	(0.04)	(0.06)	(0.03)
Exchange rate	0.10	0.02	−0.38	−0.04	−0.10	−0.07	−0.07	−0.01	−0.01	−0.05	−0.14	0.03
volatility	(0.06)	(0.02)	(0.10)	(0.02)	(0.05)	(0.04)	(0.05)	(0.04)	(0.06)	(0.05)	(0.07)	(0.04)
Stock	0.15	0.10	0.09	0.06	0.01	0.00	0.07	0.03	−0.05	0.10	0.15	0.10
volatility	(0.06)	(0.02)	(0.08)	(0.02)	(0.04)	(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.06)	(0.04)
Spread	−1.53	56.84	16.29	68.73	3.46	22.51	3.65	25.82	3.98	19.35	3.22	35.60
	(2.69)	(2.36)	(2.78)	(2.67)	(1.29)	(1.57)	(1.20)	(1.81)	(2.13)	(1.43)	(1.68)	(2.53)
Close ask	—	−0.06	0.00	0.20	—	−0.28	0.11	0.07	—	−0.02	0.02	0.05
depth	—	(0.02)	(0.10)	(0.02)	—	(0.04)	(0.06)	(0.04)	—	(0.04)	(0.08)	(0.05)
Far ask	−0.28	—	—	—	−0.21	—	—	—	−0.43	—	—	—
depth	(0.08)	—	—	—	(0.06)	—	—	—	(0.08)	—	—	—
Close bid	−0.23	0.14	—	−0.08	0.09	0.20	—	−0.08	0.02	0.17	—	−0.20
depth	(0.07)	(0.02)	—	(0.02)	(0.05)	(0.03)	—	(0.04)	(0.07)	(0.04)	—	(0.04)

(continued)

Table V—*Continued*

	BHO				ERR				WEM			
	Sell Orders		Buy Orders		Sell Orders		Buy Orders		Sell Orders		Buy Orders	
Variable	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick
Far bid	—	—	0.05	—	—	—	−0.26	—	—	—	−0.49	—
depth	—	—	(0.10)	—	—	—	(0.06)	—	—	—	(0.07)	—
Order	−0.19	−0.22	−0.50	−0.29	−0.12	−0.02	−0.25	−0.19	−0.20	−0.13	−0.30	−0.20
quantity	(0.08)	(0.03)	(0.10)	(0.03)	(0.05)	(0.03)	(0.06)	(0.04)	(0.08)	(0.05)	(0.07)	(0.04)
Recent	0.03	0.04	0.03	0.04	0.01	0.02	0.06	0.01	0.04	0.04	0.06	0.04
trades	(0.01)	(0.00)	(0.01)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Lagged	−0.14	−0.08	−0.07	−0.06	−0.15	−0.09	−0.10	−0.08	−0.08	−0.06	−0.07	−0.04
duration	(0.03)	(0.01)	(0.03)	(0.01)	(0.02)	(0.02)	(0.02)	(0.01)	(0.02)	(0.01)	(0.02)	(0.01)
Mid-quote	5.61	3.33	−2.17	−4.31	4.05	3.68	−2.53	−4.96	0.28	12.89	−5.37	−10.44
volatility	(0.80)	(0.24)	(1.08)	(0.27)	(0.47)	(0.32)	(0.70)	(1.08)	(0.91)	(1.14)	(1.18)	(1.18)
Distance to	26.97	38.51	−21.66	−45.17	36.36	49.49	−45.87	−101.78	25.43	42.53	−31.16	−81.30
mid-quote	(2.76)	(1.83)	(3.45)	(1.98)	(4.65)	(4.64)	(4.46)	(6.04)	(2.62)	(3.72)	(5.15)	(5.88)
First hour	−0.68	−0.57	0.08	−0.21	−0.55	−0.35	−0.29	0.06	−0.92	−0.81	0.05	−0.20
	(0.27)	(0.09)	(0.39)	(0.09)	(0.20)	(0.12)	(0.24)	(0.13)	(0.22)	(0.15)	(0.31)	(0.16)
Second hour	−0.20	−0.63	−0.23	−0.11	−0.55	−0.48	−0.20	−0.17	−1.03	−0.78	0.02	−0.10
	(0.27)	(0.09)	(0.39)	(0.09)	(0.20)	(0.12)	(0.24)	(0.13)	(0.22)	(0.14)	(0.32)	(0.14)
Third hour	−0.02	−0.44	−0.51	−0.26	−0.42	−0.34	−0.29	−0.13	−0.48	−0.48	0.39	−0.27
	(0.27)	(0.09)	(0.43)	(0.09)	(0.20)	(0.13)	(0.25)	(0.13)	(0.22)	(0.13)	(0.32)	(0.13)
Fourth hour	0.06	−0.41	−0.28	−0.19	−0.10	−0.11	−0.12	−0.33	−0.06	−0.44	−0.20	−0.21
	(0.28)	(0.09)	(0.44)	(0.09)	(0.20)	(0.12)	(0.25)	(0.13)	(0.21)	(0.13)	(0.34)	(0.13)
Fifth hour	0.28	−0.34	−0.46	−0.17	−0.11	−0.11	−0.10	−0.22	−0.14	−0.46	0.63	−0.19
	(0.28)	(0.09)	(0.48)	(0.09)	(0.20)	(0.13)	(0.26)	(0.14)	(0.24)	(0.14)	(0.33)	(0.14)
Sixth hour	0.40	−0.21	0.00	0.01	−0.19	−0.23	−0.17	−0.05	−0.08	−0.13	0.45	−0.07
	(0.29)	(0.09)	(0.49)	(0.09)	(0.22)	(0.13)	(0.28)	(0.13)	(0.22)	(0.13)	(0.33)	(0.13)
$\chi^2(19)$	302.61	1731.08	188.96	1710.49	290.14	531.01	403.44	539.69	218.20	501.50	244.62	493.20
p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
No. of Obs.	1,748	4,498	1,353	4,105	2,458	2,368	2,138	1,732	1,679	1,793	1,792	1,689

marginal buy limit orders is 13%, and for one-tick buy limit orders is 63%. The estimates for the other stocks are similar.

From equation (11), the picking-off risk is equal to the product of the expected change in the common value conditional on an execution and the execution probability. For buy orders, we parameterize the expected change conditional on an execution as a linear function, that is,

$$E\left[(y_{t+\tau_{execute}} - y_t) \,|\, I_t(\tau_{execute} \leq \tau_{cancel}) = 1, z_t, d_{t,b}^{buy} = 1\right] = z_t' \Lambda_b^{buy}, \qquad (49)$$

where z_t is the state vector and Λ_b^{buy} is a coefficient vector to be estimated. The sell side is treated similarly. The expectation of the change in the common value conditional on execution is determined by the trader arrival rates, the cancellation time distributions, and the future traders' order submissions. As in the case of the distribution of execution times, we use a parametric model rather than a nonparametric model to allow for a large state vector.

We estimate the model in equation (49) for buy and sell one-tick and marginal limit orders that execute in our sample using ordinary least squares. Table VI reports the estimates. The table also reports F-tests for the null hypothesis that the expected change in the common value conditional on the order executing does not depend on the state vector z_t. We reject the null hypothesis for all order submissions and stocks.

We use the parameter estimates to forecast the expected change in the common value, conditional on the limit order executing for buy and sell one-tick and marginal limit orders at every order submission. At the mean values of the state vector, the expected change in the common value is approximately zero for one-tick limit orders, minus 4 cents for marginal buy limit orders, and 4 cents for marginal sell limit orders.

We form estimates of the picking-off risk by substituting our estimates of the expected change in the common value conditional on execution and the execution probabilities into equation (11). At the mean values of the state vector, the picking-off risk is close to zero for one-tick limit orders, and approximately 1 cent for marginal limit orders.

D. Estimates of the Arrival Rates, Private Value Distributions, and Costs

We estimate the remaining parameters of the model by maximizing the conditional log-likelihood function for the timing of market and limit orders. We form the log-likelihood function for sell market orders, sell limit orders between the one-tick and the marginal sell order, buy limit orders between the one-tick and the marginal buy order, and buy market orders. The grouping is consistent with the theoretical model and leads to consistent estimators of the remaining parameters. To form the conditional log-likelihood function, we use the optimal order submission strategy, the trader arrival rates, and the distributions of the trader's private values to compute the probabilities of observing a limit order or a market order. We report the conditional log-likelihood function in Appendix D.

Table VI

Regression Model for Common Value Changes

Parameter estimates, with heteroscedasticity adjusted asymptotic standard errors in parentheses, for the regressions used to predict changes in the common value conditional on order execution. The F-test is for the null that the state vector z_t does not affect the conditional expectation, with p-values in parentheses. The three sample stocks are Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

	BHO				ERR				WEM			
	Sell Orders		Buy Orders		Sell Orders		Buy Orders		Sell Orders		Buy Orders	
Variable	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick	Marg.	1 Tick
Constant	7.30	−2.83	−6.66	−2.65	21.70	0.72	10.64	−1.21	3.21	−6.78	−11.83	−9.24
	(2.73)	(0.63)	(4.87)	(0.50)	(6.64)	(1.22)	(5.06)	(1.18)	(4.79)	(1.83)	(7.35)	(1.88)
TSX market	−0.05	−0.01	0.10	0.01	−0.14	0.07	0.21	0.03	0.28	0.15	0.10	0.03
volatility	(0.10)	(0.01)	(0.15)	(0.01)	(0.11)	(0.04)	(0.10)	(0.03)	(0.13)	(0.05)	(0.10)	(0.02)
TSX mining	0.36	0.01	−0.02	−0.01	−0.29	−0.06	−0.23	0.00	−0.10	−0.02	−0.05	−0.05
volatility	(0.12)	(0.01)	(0.16)	(0.01)	(0.13)	(0.03)	(0.09)	(0.03)	(0.14)	(0.03)	(0.14)	(0.03)
Interest rate	−0.25	−0.02	0.25	0.01	−0.10	0.05	−0.01	−0.03	0.07	0.00	−0.11	−0.03
volatility	(0.12)	(0.01)	(0.11)	(0.01)	(0.10)	(0.04)	(0.09)	(0.04)	(0.14)	(0.03)	(0.17)	(0.03)
Exchange rate	−0.02	0.01	−0.15	0.02	−0.02	−0.05	−0.07	−0.02	0.18	0.05	−0.04	0.01
volatility	(0.12)	(0.01)	(0.17)	(0.01)	(0.11)	(0.04)	(0.10)	(0.03)	(0.12)	(0.03)	(0.12)	(0.02)
Stock	−0.10	0.00	−0.26	−0.01	−0.06	0.00	0.01	−0.02	0.07	−0.06	−0.27	−0.01
volatility	(0.15)	(0.01)	(0.14)	(0.01)	(0.08)	(0.03)	(0.06)	(0.02)	(0.12)	(0.03)	(0.13)	(0.03)
Spread	3.66	−10.14	3.60	6.23	−11.32	−13.70	9.08	16.85	−15.30	−12.39	16.58	5.71
	(6.27)	(2.35)	(6.27)	(1.67)	(3.26)	(2.24)	(3.85)	(2.30)	(4.63)	(3.24)	(3.78)	(2.97)
Close ask	—	0.01	−0.08	−0.01	—	0.12	−0.12	0.05	—	0.04	0.03	−0.02
depth	—	(0.02)	(0.20)	(0.01)	—	(0.05)	(0.15)	(0.04)	—	(0.04)	(0.20)	(0.03)
Far ask	0.28	—	—	—	−0.03	—	—	—	0.31	—	—	—
depth	(0.16)	—	—	—	(0.14)	—	—	—	(0.14)	—	—	—
Close bid	−0.10	0.00	—	0.01	−0.13	−0.17	—	−0.06	−0.30	−0.12	—	0.01
depth	(0.12)	(0.01)	—	(0.01)	(0.12)	(0.04)	—	(0.04)	(0.14)	(0.04)	—	(0.03)
Far bid	—	—	−0.26	—	—	—	0.23	—	—	—	−0.07	—
depth	—	—	(0.19)	—	—	—	(0.16)	—	—	—	(0.17)	—

Order quantity	0.01	−0.03	0.21	0.00	−0.18	0.00	0.21	0.15	−0.17	−0.05	0.19	0.01
	(0.15)	(0.01)	(0.23)	(0.02)	(0.13)	(0.04)	(0.13)	(0.04)	(0.17)	(0.05)	(0.18)	(0.03)
Recent trades	0.00	0.00	0.02	0.00	0.00	−0.00	0.00	0.00	−0.03	−0.00	0.07	0.00
	(0.00)	(0.01)	(0.02)	(0.01)	(0.03)	(0.01)	(0.03)	(0.01)	(0.02)	(0.01)	(0.02)	(0.01)
Lagged duration	−0.06	0.00	0.16	0.00	−0.03	−0.02	0.05	−0.01	−0.03	−0.00	0.02	0.01
	(0.03)	(0.00)	(0.03)	(0.00)	(0.05)	(0.01)	(0.04)	(0.01)	(0.02)	(0.01)	(0.03)	(0.01)
Mid-quote volatility	−2.34	1.79	1.53	1.47	−8.85	−0.09	−9.08	0.62	0.10	4.24	4.85	5.42
	(1.42)	(0.34)	(3.00)	(0.30)	(3.78)	(0.71)	(3.18)	(0.70)	(2.80)	(1.07)	(4.44)	(1.11)
Distance to midquote	−57.05	−7.50	−42.05	−5.31	−128.71	−33.59	−149.04	−45.03	−54.86	−20.18	−55.17	−11.24
	(6.72)	(1.32)	(12.23)	(0.95)	(16.58)	(5.50)	(18.28)	(5.59)	(14.63)	(5.80)	(22.80)	(4.55)
First hour	−0.62	−0.12	0.81	−0.03	−0.84	−0.12	−0.21	−0.26	0.56	−0.02	−1.17	−0.13
	(0.70)	(0.04)	(1.01)	(0.04)	(0.60)	(0.16)	(0.50)	(0.14)	(0.47)	(0.16)	(0.54)	(0.12)
Second hour	0.20	−0.03	0.77	−0.01	−0.72	−0.31	−0.05	−0.33	0.72	−0.08	−1.04	−0.06
	(0.66)	(0.05)	(1.01)	(0.04)	(0.60)	(0.15)	(0.50)	(0.13)	(0.47)	(0.14)	(0.55)	(0.09)
Third hour	−0.05	0.03	0.87	0.04	−0.55	0.01	−0.51	−0.10	0.77	−0.24	−0.63	−0.06
	(0.66)	(0.04)	(0.98)	(0.04)	(0.57)	(0.15)	(0.52)	(0.13)	(0.42)	(0.13)	(0.50)	(0.09)
Fourth hour	−0.63	−0.02	0.89	0.01	−0.59	0.17	−0.41	−0.10	0.54	−0.21	0.15	0.08
	(0.64)	(0.04)	(1.10)	(0.04)	(0.61)	(0.16)	(0.51)	(0.12)	(0.44)	(0.14)	(0.58)	(0.07)
Fifth hour	−0.06	−0.00	0.47	0.01	−0.36	−0.36	−0.41	−0.30	−0.31	−0.03	0.07	0.02
	(0.66)	(0.04)	(1.02)	(0.04)	(0.59)	(0.15)	(0.52)	(0.12)	(0.46)	(0.14)	(0.55)	(0.08)
Sixth hour	−0.56	0.01	0.08	0.06	−0.44	0.12	−0.52	−0.11	0.20	−0.26	−0.37	−0.03
	(0.69)	(0.04)	(1.04)	(0.04)	(0.60)	(0.15)	(0.69)	(0.13)	(0.42)	(0.12)	(0.64)	(0.07)
R^2	0.33	0.13	0.44	0.09	0.28	0.14	0.40	0.20	0.35	0.20	0.29	0.09
F-test	7.64	12.45	8.64	6.90	5.84	6.21	6.58	8.66	6.77	4.08	6.39	3.45
p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
No. of Obs.	293	2,388	170	2,261	565	1,194	484	955	327	892	281	858

The conditional probability of a buy market order between t and $t + dt$ is the probability that a trader who arrives finds it optimal to submit a buy market order times the probability that a trader arrives. Using the conditional distribution of the private values, $G(u \mid z_t)$, the trader arrival rate, and the assumption that the one tick buy limit order is an optimal order submission for some trader, we have

$$\begin{aligned}\Pr(\text{Buy market order in } [t, t+dt) \mid z_t) &= \Pr\big(y_t + u_t \geq \theta_{0,1}^{buy}(z_t) \,\big|\, z_t\big)\lambda(t; x_t)\,dt \\ &= \big[1 - G\big(\theta_{0,1}^{buy}(z_t) - y_t \,\big|\, x_t\big)\big]\lambda(t; x_t)\,dt.\end{aligned} \tag{50}$$

Similarly, the probability of a buy limit order is

$$\begin{aligned}&\Pr(\text{Buy limit order in } [t, t+dt) \mid z_t) \\ &\quad = \big[G\big(\theta_{0,1}^{buy}(z_t) - y_t \,\big|\, x_t\big) - G\big(\theta_{marginal}^{buy}(z_t) - y_t \,\big|\, x_t\big)\big]\lambda(t; x_t)\,dt.\end{aligned} \tag{51}$$

The probability of a sell market order and a sell limit order are computed similarly.

We may observe no orders submitted between t and $t + dt$ for two reasons. Either a trader does not arrive, or a trader arrives and chooses not to submit any order, so that

$$\begin{aligned}&\Pr(\text{No order submission in } [t, t+dt) \mid z_t) \\ &= 1 - \lambda(t; x_t)\,dt + \big[G\big(\theta_{marginal}^{buy}(z_t) - y_t \,\big|\, x_t\big) - G\big(\theta_{marginal}^{sell}(z_t) - y_t \,\big|\, x_t\big)\big]\lambda(t; x_t)\,dt.\end{aligned} \tag{52}$$

The conditional probabilities in equations (50)–(52) depend on the trader arrival rates, $\lambda(t; x_t)$, the private values distribution, $G(u \mid x_t)$, and the threshold values, that is, the θs. The threshold values themselves depend on the costs, the execution probabilities, and the picking-off risks—see equations (16)–(20). We substitute our first-step estimates of the execution probabilities and picking-off risks into the threshold functions to form the log-likelihood function.

Let t_i be the time of the i^{th} order arrival. We use a Weibull parameterization for the trader arrival rate. Suppose that the last order submission was at time t_{i-1}. The arrival rate is

$$\lambda(t; x_{t_i})\,dt = \exp(x_{t_i}'\delta)\eta(t - t_{i-1})^{\eta-1}\,dt, \tag{53}$$

where x_{t_i} denotes the exogenous state variables immediately before the order submission at time t_i. The private value distribution is parameterized as a mixture of two normal distributions with standard deviations depending on the common value and exogenous state variables,

$$G(u \mid x_t) = \rho\Phi\left(\frac{u}{y_t\sigma_1\exp(x_t'\Gamma)}\right) + (1-\rho)\Phi\left(\frac{u}{y_t\sigma_2\exp(x_t'\Gamma)}\right), \tag{54}$$

where Φ denotes the standard normal cumulative distribution function, the parameter satisfies $\sigma_1 \neq \sigma_2$, and $0 < \rho < 1$, and x_t denotes the exogenous state variables. Because the standard deviation is proportional to the common value, the private values are normalized as percentages of the common value.

Table VII reports the conditional maximum likelihood estimates of parameters of the arrival rates and traders' private values and execution costs, c_e, with standard errors reported in parentheses. The likelihood function is relatively flat with respect to the order submission cost, for positive order submission costs. We therefore do not estimate the order submission cost, c_o, but set it equal to 0.1% of the common value.

The first row of the top panel reports estimates of the Weibull shape parameter, η. The parameter estimate is less than one for all stocks: The longer the time since the last order submission, the lower the conditional probability that a new trader will arrive. The second through seventh rows of the top panel report estimates of the parameters on the exogenous state variables. The exogenous state variables are standardized by dividing by their sample standard deviations.

The estimated parameters on many of the exogenous state variables are positive. The arrival rate of traders increases following periods of high TSX market volatility. The parameters on stock volatility are positive and larger in magnitude than the parameters on the other exogenous state variables. The relative magnitudes of the parameters suggest that stock-specific shocks are more important than market-wide shocks for the trader arrival rates. Higher stock volatility predicts an increase in the trader arrival rates.

The second panel reports parameter estimates for the distributions of the private values. For all stocks, the private values distribution is a mixture of two normal distributions, with approximately 85% weight on a distribution with a standard deviation of approximately 5% and a 15% weight on a distribution with a standard deviation of approximately 54%. The point estimates of the order execution costs, c_e, are reported in the third panel, and are between 1% and 2% of the common value.

The estimated parameters on many of the exogenous state variables are statistically significant. A change in stock volatility has the largest effect on the distributions of the private values. The parameters on stock volatility are all negative; when stock volatility is high, a higher fraction of the traders that arrive have private values close to zero. One interpretation of this relation is persistence in stock volatility: If high stock volatility is often associated with past high stock volatility and high volume, then perhaps the traders with the most extreme private values have already altered their portfolios so that the traders left have less extreme private values.

Our estimates assume that the exogenous variables follow a strictly Markov process. Some of the results may change with the addition of additional lags. We believe that our estimates of the gains from trade are likely to be robust to the inclusion of additional lags of the exogenous state variables, however.

Table VIII reports the expected utilities for traders with six different private values across three different market conditions, namely, a low liquidity

Table VII

Conditional Maximum Likelihood Estimates

The table reports estimates of the trader arrival rate, the private value distribution and the trading opportunity cost. Asymptotic standard errors are reported in parentheses. The table also reports χ^2 tests for the null hypothesis of a constant trader arrival rate and a constant private value variance, with p-values in parentheses. The three sample stocks are Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

	BHO	ERR	WEM
Trader Arrival Rate			
Shape	0.45	0.47	0.45
	(0.00)	(0.00)	(0.00)
Constant	−2.15	−2.08	−2.55
	(0.02)	(0.02)	(0.02)
TSX market volatility	0.03	0.02	0.03
	(0.01)	(0.01)	(0.01)
TSX mining volatility	0.03	−0.00	0.01
	(0.01)	(0.01)	(0.01)
Interest rate volatility	−0.11	−0.00	0.07
	(0.01)	(0.01)	(0.01)
Exchange rate volatility	0.11	0.06	−0.04
	(0.01)	(0.01)	(0.01)
Stock volatility	0.27	0.15	0.19
	(0.01)	(0.01)	(0.01)
Private Value Distribution			
Mixing probability, ρ	0.85	0.89	0.84
	(0.04)	(0.03)	(0.05)
σ_1	0.06	0.04	0.04
	(0.00)	(0.00)	(0.00)
$\sigma_2 - \sigma_1$	0.47	0.68	0.48
	(0.04)	(0.06)	(0.05)
Time-Varying Variance			
TSX market volatility	−0.00	0.03	−0.05
	(0.01)	(0.01)	(0.01)
TSX mining volatility	−0.03	−0.02	−0.05
	(0.01)	(0.01)	(0.01)
Interest rate volatility	0.07	−0.05	−0.03
	(0.01)	(0.01)	(0.01)
Exchange rate volatility	−0.04	−0.02	0.03
	(0.01)	(0.01)	(0.01)
Stock volatility	−0.10	−0.14	−0.12
	(0.01)	(0.01)	(0.01)
Order execution cost, c_e	0.02	0.02	0.01
	(0.00)	(0.00)	(0.00)
$\chi^2(5)$: Constant arrival rate	3871.95	680.94	985.28
	(0.00)	(0.00)	(0.00)
$\chi^2(5)$: Constant private value variance	434.55	814.08	369.70
	(0.00)	(0.00)	(0.00)

Table VIII

Expected Utility by Trader Valuation

The table reports, for traders with different private values, the expected utilities for alternative order submissions. The expected utilities are computed for three different states, namely, a low liquidity state in which the bid–ask spread is one standard deviation above its mean and the depth variables are one standard deviation below their means, a high liquidity state in which the bid–ask spread is one standard deviation below its mean and the depth variables are one standard deviation above their mean values, and a moving market state in which the common value proxy divided by the mid-quote is one standard deviation above its mean. All other variables are at their in-sample mean values. All utilities are reported in cents per share. The maximum utility is indicated for each private value and market condition with a box. The three sample stocks are Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

	BHO Private Value						ERR Private Value						WEM Private Value					
Percent	−5.00	−2.50	−1.25	1.25	2.50	5.00	−5.00	−2.50	−1.25	1.25	2.50	5.00	−5.00	−2.50	−1.25	1.25	2.50	5.00
Cents	−4.25	−2.12	−1.06	1.06	2.12	4.25	−9.68	−4.84	−2.42	2.44	4.84	9.68	−6.75	−3.38	−1.69	1.69	3.38	6.75
Low liquidity state: Wide spread, low depth																		
Buy market	−6.31	−4.19	−3.13	−1.00	0.06	2.18	−13.65	−8.81	−6.39	−1.55	0.87	5.71	−9.94	−6.57	−4.88	−1.51	0.18	3.55
Buy 1 tick	−4.33	−2.45	−1.51	0.38	1.32	**3.20**	−8.99	−5.08	−3.13	0.77	2.73	**6.63**	−7.12	−4.26	−2.84	0.02	1.45	**4.31**
Buy marginal	1.31	1.54	1.65	**1.88**	**2.00**	2.22	3.46	3.93	4.16	**4.63**	**4.86**	5.33	0.97	1.27	1.42	**1.72**	**1.87**	2.17
No order	1.79	1.79	1.79	1.79	1.79	1.79	4.61	4.61	4.61	4.61	4.61	4.61	1.54	1.54	1.54	1.54	1.54	1.54
Sell marginal	2.47	**2.04**	**1.83**	1.40	1.18	0.75	5.72	**5.02**	**4.66**	3.96	3.61	2.90	2.67	**2.11**	**1.83**	1.28	1.00	0.44
Sell 1 tick	**3.16**	1.28	0.35	−1.52	−2.46	−4.33	**6.17**	2.05	0.00	−4.12	−6.17	−10.29	**3.64**	1.64	0.64	−1.36	−2.36	−4.36
Sell market	2.18	0.06	−1.00	−3.13	−4.19	−6.31	4.56	−0.28	−2.70	−7.54	−9.96	−14.80	3.56	0.18	−1.51	−4.88	−6.57	−9.95
High liquidity state: Narrow spread, high depth																		
Buy marginal	−4.36	−2.24	−1.18	0.95	2.01	**4.13**	−11.70	−6.86	−4.44	0.40	2.82	**7.66**	−7.23	−3.86	−2.17	1.20	**2.89**	**6.26**
Buy 1 tick	−0.27	0.55	0.95	1.77	**2.17**	2.99	1.28	2.35	2.89	3.97	4.51	5.58	−0.91	0.11	0.62	**1.64**	2.15	3.17
Buy marginal	1.38	1.54	1.61	1.77	1.84	2.00	3.37	3.84	4.07	4.54	**4.78**	5.25	1.11	1.28	1.37	1.55	1.64	1.81
No order	1.79	1.79	**1.79**	**1.79**	1.79	1.79	4.61	4.61	**4.61**	**4.61**	4.61	4.61	1.54	1.54	1.54	1.54	1.54	1.54
Sell marginal	2.12	1.88	1.76	1.51	1.39	1.15	5.40	**4.80**	4.50	3.91	3.61	3.02	2.28	1.85	1.63	1.20	0.99	0.55
Sell 1 tick	2.81	**2.05**	1.67	0.91	0.53	−0.23	5.23	4.48	4.10	3.35	2.97	2.22	3.94	2.48	**1.75**	0.29	−0.44	−1.90
Sell market	**4.13**	2.01	0.95	−1.18	−2.24	−4.36	**8.81**	3.97	1.55	−3.29	−5.71	−10.55	**6.27**	**2.89**	1.20	−2.17	−3.86	−7.23
Moving market state: Common value proxy higher than the mid-quote																		
Buy market	−3.69	−1.52	−0.44	1.73	2.81	**4.98**	−5.12	−0.08	2.44	**7.48**	**10.00**	**15.04**	−6.04	−2.60	−0.87	**2.57**	**4.29**	**7.73**
Buy 1 tick	−0.77	0.44	1.04	**2.25**	**2.86**	4.07	3.72	4.12	4.32	4.72	4.92	5.32	−0.77	0.38	0.96	2.11	2.68	3.83
Buy marginal	1.50	1.63	1.70	1.83	1.90	2.03	4.24	4.38	4.45	4.58	4.65	4.79	1.25	1.40	1.47	1.62	1.70	1.84
No order	1.79	1.79	**1.79**	1.79	1.79	1.79	4.61	4.61	**4.61**	4.61	4.61	4.61	1.54	1.54	**1.54**	1.54	1.54	1.54
Sell marginal	**2.50**	**1.97**	1.71	1.18	0.92	0.40	**6.22**	**4.65**	3.87	2.30	1.52	−0.05	2.53	**1.85**	1.51	0.83	0.49	−0.19
Sell 1 tick	2.35	0.77	−0.03	−1.61	−2.40	−3.99	2.48	−1.08	−2.86	−6.42	−8.20	−11.75	**2.84**	0.99	0.06	−1.78	−2.71	−4.56
Sell market	1.51	−0.66	−1.74	−3.91	−4.99	−7.16	−0.87	−5.91	−8.43	−13.47	−15.99	−21.03	2.36	−1.08	−2.81	−6.25	−7.97	−11.41

state with a wide spread and low depth, a high liquidity state with a narrow spread and high depth, and a moving market state where the common value is above the mid-quote. For each private value, we report the expected utility from submitting a market order, a one-tick limit order, a marginal limit order, or no order at all. The private values are 1.25%, 2.5%, or 5% higher or lower than the common value; the corresponding private values measured in cents are reported in the second row. The reported expected utility is a lower bound on the trader's expected utility because we do not compute the expected utility for limit orders between one-tick from the quotes and marginal limit orders. The maximum utility is indicated for each private value and market condition with a box.

In the low liquidity state, a trader with a private value equal to 2.5% or −2.5% optimally submits a marginal limit order, and a trader with private value equal to 5% or −5% optimally submits a one-tick limit order. In the high liquidity state, a trader with a private value equal to 1.25% or −1.25% optimally submits no order in BHO and Eurus Resources (ERR), and optimally submits a limit order in War Eagle Mining Company (WEM). A trader with a private value equal to 2.5% or −2.5% optimally submits limit orders for BHO and ERR and submits market orders for WEM. A trader with a private value equal to 5% or −5% optimally demands liquidity by submitting a market order for all stocks.

In the moving market state, the optimal order strategies are asymmetric. With a high common value relative to the mid-quote, the optimal strategy is to submit market orders in ERR and WEM for traders with all three positive private values. Such market order submissions pick off some sell limit orders. For BHO, traders with private values equal to 1.25% and 2.5% submit one-tick limit orders rather than market orders.

The expected utilities reported in Table VIII show that traders' optimal order submission strategy is state dependent: For a given private value, the optimal order submission changes with the state vector.

Table IX reports summary statistics for the estimated private value distributions and the optimal order submission strategies for five intervals for the private value. The first row in each panel reports the mean proportion of traders in each private value interval. The next rows report the mean and standard deviations of the fitted order submission probabilities for a sell market order, a sell limit order, no order, a buy limit order, and a buy market order.

Traders with extreme private values typically choose market or limit orders and rarely choose not to submit an order. Traders with intermediate private values submit limit orders most frequently but also use market orders and sometimes choose not to submit orders. Traders with private values close to zero almost always submit limit orders when they choose to submit an order, but often choose not to submit an order.

The standard deviations of the order submission probabilities reported in Table IX indicate that for all five private value intervals, the state dependence in the optimal order submission strategy is economically significant—traders' optimal order submissions change frequently as the traders' information set and the limit-order book changes.

Table IX

Order Submission Probabilities by Trader Private Value

The table reports the in-sample mean of the probability of drawing a private value (u) from five different intervals. For each interval and stock the table reports the in-sample mean and standard deviation of the probability of a trader optimally submitting a sell market order, a sell limit order, no order, a buy limit order, or a buy market order, conditional on the trader's private value being in that interval. The three sample stocks are Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

Order Submission		Private Value				
		(−∞, −5%]	(−5%, −2.5%]	(−2.5%, +2.5%)	[2.5%, +5%)	[+ 5%, +∞)
			BHO			
Mean probability of private value in interval		0.21	0.13	0.32	0.13	0.21
			Order Submission Probabilities			
Sell market	Mean	0.74	0.25	0.01	0.00	0.00
	Standard deviation	0.30	0.36	0.07	0.01	0.00
Sell limit	Mean	0.24	0.64	0.25	0.00	0.00
	Standard deviation	0.29	0.39	0.20	0.04	0.01
No order	Mean	0.02	0.09	0.36	0.07	0.01
	Standard deviation	0.09	0.25	0.28	0.22	0.07
Buy limit	Mean	0.00	0.02	0.37	0.74	0.30
	Standard deviation	0.03	0.13	0.27	0.36	0.31
Buy market	Mean	0.00	0.00	0.01	0.19	0.69
	Standard deviation	0.00	0.01	0.05	0.32	0.31
			ERR			
Mean probability of private value in interval		0.12	0.14	0.48	0.14	0.12
			Order Submission Probabilities			
Sell market	Mean	0.85	0.32	0.01	0.00	0.00
	Standard deviation	0.23	0.35	0.06	0.03	0.01
Sell limit	Mean	0.15	0.66	0.16	0.00	0.00
	Standard deviation	0.23	0.36	0.12	0.03	0.01
No order	Mean	0.00	0.03	0.61	0.01	0.00
	Standard deviation	0.03	0.12	0.15	0.07	0.02
Buy limit	Mean	0.00	0.00	0.21	0.75	0.22
	Standard deviation	0.00	0.02	0.12	0.33	0.26
Buy market	Mean	0.00	0.00	0.01	0.24	0.78
	Standard deviation	0.00	0.00	0.04	0.33	0.26
			WEM			
Mean probability of private value in interval		0.11	0.13	0.52	0.13	0.11
			Order Submission Probabilities			
Sell market	Mean	0.92	0.48	0.04	0.00	0.00
	Standard deviation	0.16	0.41	0.09	0.00	0.00
Sell limit	Mean	0.08	0.51	0.41	0.00	0.00
	Standard deviation	0.16	0.41	0.17	0.05	0.01
No order	Mean	0.00	0.01	0.12	0.01	0.00
	Standard deviation	0.01	0.05	0.18	0.09	0.02
Buy limit	Mean	0.00	0.00	0.40	0.62	0.16
	Standard deviation	0.00	0.02	0.12	0.42	0.24
Buy market	Mean	0.00	0.00	0.03	0.37	0.84
	Standard deviation	0.00	0.01	0.09	0.41	0.24

IV. Estimates of the Gains from Trade

Substituting the mixture of normal distributions assumption for the private values distribution in equation (54) into the maximum gains from trade in equation (33), the maximum gains from trade as a percentage of the common value is

$$\begin{aligned}&\text{Maximum gains}\,(x_{t_i})\\&=\rho\left(2\sigma_1\exp\left(x_{t_i}'\Gamma\right)\phi\left(\frac{c_e+c_o}{\sigma_1\exp\left(x_{t_i}'\Gamma\right)}\right)-(c_e+c_o)2\Phi\left(\frac{-c_e-c_o}{\exp\sigma_1\left(x_{t_i}'\Gamma\right)}\right)\right)\\&\quad+(1-\rho)\left(2\sigma_2\exp\left(x_{t_i}'\Gamma\right)\phi\left(\frac{c_e+c_o}{\sigma_2\exp\left(x_{t_i}'\Gamma\right)}\right)-(c_e+c_o)2\Phi\left(\frac{-c_e-c_o}{\sigma_2\exp\left(x_{t_i}'\Gamma\right)}\right)\right).\end{aligned} \tag{55}$$

Here ϕ is the standard normal density function, and Φ is the standard normal cumulative distribution function. In Appendix E we derive equation (55) and provide a description of how we compute the current gains, and the losses from no execution, no submission, wrong direction, extramarginal submission, and the monopoly gains.

The current gains from trade in the limit-order market depend on the threshold valuations and the execution probabilities for limit orders. We estimate the conditional execution probabilities for the marginal and the one-tick buy and sell limit orders. If in any given state the execution probabilities for any buy limit order are greater than or equal to the execution probability for the marginal buy limit order, then using the execution probability for the marginal buy limit order for all buy limit orders in that state provides a lower bound on the current gains from trade on the buy side. An analogous argument applies to the sell side. Similarly, using the execution probabilities for one-tick limit orders for all limit order execution probabilities provides an upper bound on the gains from trade. We report the lower and upper bounds for the current gains from trade. We also report the average current gains from trade, that is, the gains computed using the average of the one-tick and marginal buy or sell execution probabilities for all buy or sell limit orders in a given state.

We take expectations across states z_{t_i} and report the unconditional expected current and maximum gains from trade. For brevity we refer to the expected gains simply as the current gains from trade.

We observe an order submission when a trader arrives and submits an order. Suppose an order is submitted at time t_i with the state vector equal to z_{t_i}. We compute the expected gains from trade at that information set, allowing us to obtain a sample of expected gains sampled at every order submission. Let $\widehat{\text{Current gains}}(z_{t_i})$ be the resulting estimate of the current gains. Some traders who arrive may find it optimal to submit no order, implying that our sample is drawn conditional on observing an order submission.

Let I be the total number of observations in our sample. Because we only compute gains when we observe an order submission, the sample average

$$\frac{1}{I}\sum_{i=1}^{I}\widehat{\text{Current gains}}(z_{t_i}) \tag{56}$$

is an estimate of

$$E[\text{Current gains}(z_t)\,|\,\text{Trader arrives and submits and order}]. \tag{57}$$

But this measure misses the gains from trade when a trader arrives at the market but chooses not to submit an order. To include such cases, we wish to measure the gains from trade conditional on a trader arriving,

$$E[\text{Current gains}(z_t)\,|\,\text{Trader arrives}]. \tag{58}$$

We reweight our sample to estimate equation (58). In Appendix F we show that

$$\Pr(z_t\,|\,\text{Trader arrives}) = \Pr(z_t\,|\,\text{Trader arrives and submits an order}) \times \omega(z_t), \tag{59}$$

with

$$\omega(z_t) = \frac{\Pr(\text{Trader submits an order}\,|\,\text{Trader arrives})}{\Pr(\text{Trader submits an order}\,|\,z_t, \text{Trader arrives})}. \tag{60}$$

From equations (59) and (60), we have

$$E[\text{Current gains}(z_t) \times \omega(z_t)\,|\,\text{Trader arrives and submits and order}] = E[\text{Current gains}(z_t)\,|\,\text{Trader arrives}]. \tag{61}$$

We describe how we use our first-stage and maximum likelihood estimates to obtain consistent estimates of $\omega(z_{t_i})$ in Appendix F. We use the sample average

$$\frac{1}{I}\sum_{i=1}^{I}\widehat{\text{Current gains}}(z_{t_i}) \times \hat{\omega}(z_{t_i}) \tag{62}$$

to form a consistent estimate of $E[\text{Current gains}(z_t)\,|\,\text{Trader arrives}]$. Our estimates of the maximum gains from trade, the losses from no execution, no submission, wrong direction, and extramarginal submission, and the monopoly gains are all weighted in the same way.

The top panel of Table X reports estimates of the maximum and the current gains from trade. The gains are reported as a percentage of the common value. The first row reports the maximum gains from trade. In the next three rows we report estimates of the lower and upper bounds for the current gains from trade and the average current gains from trade. In rows five through seven we report the difference between the maximum and the current gains from trade and in the last three rows of the top panel we report the current gains from trade as a percentage of the maximum gains from trade.

Across the three stocks, the average of the maximum gains is approximately 8.2% of the common values and the average of the lower bound on the current

Table X

Estimates of the Gains from Trade

The first row of the top panel reports the estimates of the maximum gains from trade, measured as a percent of the common value. The next three rows report the estimates of the lower and upper bounds and the average current gains from trade; details are provided in Appendix E. The next six rows report the lower and upper bounds and the average for the difference between the maximum and the current gains from trade, and the current gains from trade as a percentage of the maximum gains from trade. The middle panel reports the average percentage of the efficiency loss associated with no execution, no submission, wrong direction, and extramarginal submissions computed for the average current gains from trade. The bottom panel reports the monopoly gains from trade as a percentage of the common value, and the current gains as a percentage of the monopoly gains. We verify that the second-order conditions hold at the computed monopoly bid and ask quotes at each observation. The three sample stocks are Barkhor Resources (BHO), Eurus Resources (ERR), and War Eagle Mining Company (WEM).

	BHO	ERR	WEM
	Gains		
	Maximum gains as a % of the common value		
	9.07	8.61	6.75
	Current gains as a % of the common value		
Lower bound	7.88	8.09	6.08
Upper bound	8.45	8.31	6.40
Average	8.16	8.20	6.24
	Maximum gains minus current gains		
Lower bound	0.62	0.30	0.35
Upper bound	1.20	0.52	0.67
Average	0.91	0.41	0.51
	Current gains as a % of maximum gains		
Lower bound	86.79	93.97	90.07
Upper bound	93.13	96.57	94.81
Average	89.96	95.27	92.44
	Decomposition of Losses		
	No execution as a % of total losses		
Sell side	32.32	31.20	33.05
Buy side	40.10	39.01	41.85
Subtotal	72.42	70.21	74.90
	No submission as a % of total losses		
Sell side	2.24	0.62	0.41
Buy side	1.98	0.15	0.71
Subtotal	4.22	0.77	1.12
	Wrong direction as a % of total losses		
Sell side	0.86	0.02	0.39
Buy side	0.20	0.05	0.63
Subtotal	1.06	0.07	1.02
	Extramarginal submissions as a % of total losses		
Sell side	9.81	11.87	10.30
Buy side	12.49	17.07	12.66
Subtotal	22.30	28.94	22.96
Total	100.00	100.00	100.00
	Monopoly Gains		
	Monopoly gains as a % of the common value		
	5.02	5.57	4.18
	Monopoly gains as a % of maximum gains		
	55.31	64.71	61.87
	Current gains as a % of monopoly gains		
	162.65	147.23	149.41

gains is around 7.4% of the common values. The maximum benefit of increasing the efficiency of the allocations resulting from current market design—the limit-order market—is less than 1% of the stocks' common values. The current gains from trade are approximately 90% of the maximum gains from trade. We interpret the magnitudes as evidence that in our sample, the limit-order market allows the traders to achieve a relatively efficient stock allocation.

From Table I, approximately 60% of the order submissions in our sample are limit orders. The average fitted execution probability for limit orders is approximately 40% across the three stocks, implying that approximately 36% of the order submissions result in no execution. If we ignore the valuations of the traders, the order submissions and fitted execution probabilities imply that the current gains from trade are approximately 64% of the maximum gains from trade. But we estimate the current gains from trade to be approximately 90% of the maximum gains from trade. We estimate the current gains from trade higher than 64% because the traders who contribute the most to the gains from trade endogenously submit market orders and thereby avoid the risk of no execution altogether, whereas the traders who contribute less to the gains from trade endogenously bear most of the risk of no execution. For example, from Table IX, approximately 80% of the traders with valuations more than 5% away from the common value submit market orders, while approximately 65% of the traders with valuations between 2.5% and 5% away from the common value submit limit orders.

In our model, losses do not arise only from unexecuted order submissions. Losses also arise from no submission, wrong direction, and extramarginal submission. The bottom panel of the table reports the decomposition of the average losses as percentages of the total losses. No execution is the most important sources of losses, accounting for between 70% and 75% of the losses.

No submission accounts for between 1% and 4% of the losses. Using the order submission cost of 0.1% of the common value and the order execution cost of approximately 2%, all traders with private values farther than 2.1% from the common value should submit orders that execute. From Table IX, less than 3% of the traders with valuations greater than 2.5% away from the common value do not submit an order.

Wrong direction accounts for no more than 1.1% of the losses. The result suggests that it is rare to see extremely stale limit orders that entice high private value traders to sell or low private value traders to buy.

Extramarginal submission accounts for between 22% and 29% of the losses. Using the order submission cost of 0.1% of the common value and the trading cost of approximately 2%, all traders with private values less than 2.1% away from the common value should not submit orders. From Table IX, approximately 64% of the traders with valuations less than 2.5% away from the common value submit an order. The losses from extramarginal submission are smaller than 64% for two reasons. First, the cut-off of 2.5% includes some traders who should trade. Second, all traders who submit orders that lead to losses from extramarginal submissions have valuations that are less extreme than the valuations of traders who should trade.

The traders in our model solve a one-shot order submission problem and do not have an opportunity to return to the market. In reality, however, some of the order submissions that lead to wrong direction or extramarginal submission may be reversed. Such trades that are later reversed may provide liquidity in a speculative fashion and thereby improve efficiency. Our estimates of the current gains may therefore be biased downward. Because wrong direction trades only account for 1% of the losses and extramarginal submissions account for about 25% of the losses, the downward bias may be small, however. To the extent our estimates are downward biased, then our estimates that the current limit-order market leads to the traders realizing 90% of the maximum gains is a lower bound.

The bottom panel of Table X reports estimates of the monopoly gains from trade. Across the three stocks, the average of the monopolist gains is approximately 5% of the common value, and 60% of the maximum gains. The monopolist maximizes profits by charging a wide spread, which reduces trading relative to the trading that leads to the allocation that maximizes the gains from trade.

We find that the current gains are approximately 150% of the monopoly gains. Thus, the current limit-order system leads to a much more efficient allocation of the stock than the monopolist generates. Most of the losses in the current limit-order market relative to the gains in the efficient allocation arise from nonexecution. Although the monopolist guarantees execution at his posted quotes, his profit-maximizing spread is large enough to significantly reduce the trading relative to trading that leads to the allocation in the current limit-order market. As a consequence, the monopoly gains are significantly lower than the gains in the current limit-order market.

A critical assumption for the counterfactual calculations of the maximum gains and the monopoly gains is that that the underlying arrival rate of traders, λ, as well as the distribution of the traders' private valuations, G, are invariant to the changes in the trading rules. If changes in trading rules have feedback effects on the arrival rate or the distribution of private valuations then the resulting gains from trade are also likely to change. Notwithstanding such issues, our gains calculations provide a good starting point for evaluating the design of the limit-order market.

Overall, our empirical work provides evidence that the limit-order market allows traders to realize many of the potential gains from trade, and significantly more gains than would be achieved under a profit-maximizing monopolist that provides bid and ask quotes.

V. Conclusions

In a perfectly liquid market in which the traders can realize the maximum gains from trade, the market's design provides incentives to traders with high private values to buy the stock, traders with low private values to sell the stock, and traders with intermediate private values—traders who have no strong need to trade—to abstain from trading. We compute the expected current, maximum, and monopolist gains from trade using a sample of both order submissions and execution and cancellation histories from a limit-order market. The current gains are approximately 90% of the maximum gains from trade and 150% of the monopolist gains from trade. The limit-order market therefore allows the traders to realize many of the gains from trade.

Close to three-quarters of the difference between the current and maximum gains from trade arise from nonexecution. Nonexecution arises because the limit-order market provides incentives for traders to submit orders with high execution risk. Almost one quarter of the difference between the current and maximum gains comes from extramarginal submissions. Extramarginal submissions occur when traders with no strong need to trade find profitable trading opportunities.

A key feature of our model is that we base our computations of the gains from trade on estimates of the traders' optimal order submission strategy in the limit-order market. Our estimates condition on the endogeneity of traders' order submissions. Such endogeneity is consistent with empirical evidence that the composition of the order flow changes as the limit-order book and market conditions change. Traders with more extreme private values tend to submit orders with low execution risk and low picking-off risk and traders with more moderate private values tend to submit limit orders with higher execution risk and higher picking-off risk. Our estimates are also consistent with the ability of traders to switch between market orders and limit orders when the relative payoffs of market and limit orders change. Both effects are important empirically.

Because our empirical approach builds directly on the theoretical model, our parameter estimates can be used in numerical solutions for the equilibrium and allow researchers to study efficiency under alternative trading rules. Goettler, Parlour, and Rajan (2005) use numerical methods to solve for the equilibrium of a model similar to ours. They determine the impact of a change in the tick size for a model using a normal distribution for the private values with a standard deviation of $\$\frac{1}{4}$ and zero costs. They report the current gains as \$0.1693 in the $\$\frac{1}{8}$-tick regime and \$0.1728 in the $\$\frac{1}{16}$-tick regime. Plugging these figures into equation (55), the maximum gains from trade are \$0.1995; the current gains are 84.9% of the maximum gains in the $\$\frac{1}{8}$-tick regime and 86.6% in the $\$\frac{1}{16}$-tick regime.

Our model is based on a one-shot order submission problem with no endogenous order cancellations or order resubmissions. Much work on the efficiency of double auctions argues that double auctions are relatively efficient because the traders can continue to resubmit bids until many of the gains from trade are exhausted. It would therefore be interesting to extend our model and empirical approach to allow for the possibility of endogenous cancellations and resubmissions. The theoretical model in Rosu (2004) features endogenous cancellations and resubmissions in a somewhat different setting from ours. His model could provide a starting point for such empirical work. Allowing for endogenous cancellations and order resubmissions may increase our estimates of the efficiency of the limit-order market.

Our model also takes as exogenous the order submission quantity. However, a trader is likely to choose how many shares to buy or sell depending on his marginal valuations for additional shares. Studying the robustness of our empirical findings to quantity dependent valuations and endogenous quantity choice would be a useful direction for future research.

Some limit-order markets have added designated market makers who have an obligation to submit limit orders in less actively traded stocks. The designated market makers typically pay lower order submission fees than other traders, but the market makers face the same rules as all other traders submitting orders. For example, Euronext Paris has designated market makers for small and less actively traded companies (Mann, Venkataraman, and Waisburd (2002)). In order to evaluate whether adding the designated market makers leads to a net efficiency improvement of the limit-order market, we need to be able to measure the gains from trade with and without designated market makers. Our method is well suited to generating such measurements.

Appendix A: Proofs

Proof of Lemma 1: The objective function in equation (29) is increasing in the private value u for the traders who buy the stock in the optimal allocation: If a trader with private value u buys the stock in the optimal allocation, then a trader with valuation $u' > u$ also buys the stock in the optimal allocation. A similar result holds for traders who sell the stock in the optimal allocation. There therefore exist two numbers, $L(x_t)$ and $H(x_t)$, such that the optimal allocation is

$$I^{sell*}(u_t; x_t) = \begin{cases} 1, & u_t \leq L(x_t), \\ 0, & \text{otherwise} \end{cases}, \quad I^{buy*}(u_t; x_t) = \begin{cases} 1, & u_t \geq H(x_t), \\ 0, & \text{otherwise} \end{cases}. \tag{A1}$$

Let $P(x_t)$ be a solution for P to

$$G(P - c_e - c_o \mid x_t) - (1 - G(P + c_e + c_o \mid x_t)) = 0. \tag{A2}$$

As P goes to $-\infty$, $G(P - c_e - c_o \mid x_t)$ goes to zero and $(1 - G(P + c_e + c_o \mid x_t))$ goes to one; and as P goes to ∞, $G(P - c_e - c_o \mid x_t)$ goes to one and $(1 - G(P + c_e + c_o \mid x_t))$ goes to zero. The function in equation (A2) is monotonic in P and by the continuity of the distribution, G, the function is also continuous in P. A unique solution therefore always exists to equation (A2). If the distribution is symmetric, $P(x_t)$ is the median of the distribution.

We now show that the cut-offs

$$L(x_t) = P(x_t) - c_e - c_o, \quad H(x_t) = P(x_t) + c_e + c_o \tag{A3}$$

characterize the optimal allocation. By construction, the market clearing condition is satisfied at the allocation. Suppose that $\hat{L} > P(x_t) - c_e - c_o$ and $\hat{H}$ characterize an alternative allocation. The market clearing condition for the alternative allocation is

$$G(\hat{L} \mid x_t) = 1 - G(\hat{H} \mid x_t). \tag{A4}$$

Equations (A2) and (A4) imply $\hat{H} < P(x_t) + c_e + c_o$ and

$$\int_{P(x_t)-c_e-c_o}^{\hat{L}} g(u \mid x_t)\, du - \int_{\hat{H}}^{P(x_t)+c_e+c_o} g(u \mid x_t)\, du = 0. \tag{A5}$$

Let $I(u_t \leq L)$ be an indicator function for $u_t \leq L$, with $I(u_t \geq H)$ defined similarly. The difference in the expected gains from trade between the two allocations is

$$\begin{aligned}
&E\big[I(u_t \leq \hat{L})(-u_t - c_e - c_o) + I(u_t \geq \hat{H})(u_t - c_e - c_o) \mid x_t\big] \\
&\quad - E[I(u_t \leq P(x_t) - c_e - c_o)(-u_t - c_e - c_o) \\
&\quad + I(u_t \geq P(x_t) + c_e + c_o)(u_t - c_e - c_o) \mid x_t] \\
&= \int_{P(x_t)-c_e-c_o}^{\hat{L}} (-u - c_e - c_o) g(u \mid x_t)\, du + \int_{\hat{H}}^{P(x_t)+c_e+c_o} (u - c_e - c_o) g(u \mid x_t)\, du \\
&< \int_{P(x_t)-c_e-c_o}^{\hat{L}} (-(P(x_t) - c_e - c_o) - c_e - c_o) g(u \mid x_t)\, du \\
&\quad + \int_{\hat{H}}^{P(x_t)+c_e+c_o} ((P(x_t) + c_e + c_o) - c_e - c_o) g(u \mid x_t)\, du \\
&= P(x_t)\left(- \int_{P(x_t)-c_e-c_o}^{\hat{L}} g(u \mid x_t)\, du + \int_{\hat{H}}^{P(x_t)+c_e+c_o} g(u \mid x_t)\, du\right) \\
&= 0,
\end{aligned} \tag{A6}$$

where the inequality follows from the monotonicity in u of both integrals.

Similar logic holds for alternative allocations with $\hat{L} < P(x_t) - c_e - c_o$. Q.E.D.

Proof of Lemma 2: Using the logic in the proof to Lemma 1, the trader's optimal decision can still be characterized by two cut-offs. A trader optimally buys if the utility from buying is greater than the utility from not trading, that is,

$$y_t + u_t - c_e - c_o - a_t^m \geq 0, \tag{A7}$$

implying the cut-off

$$u_t \geq a_t^m + c_e + c_o - y_t. \tag{A8}$$

An analogous result holds for the sell decision.

Using the distribution of private values, the monopolist's profit-maximization problem is

$$\max_{\{b_t^m, a_t^m\}} G\big(b_t^m - c_e - c_o - y_t \,\big|\, x_t\big)\big(y_t - b_t^m\big) + \big(1 - G\big(a_t^m + c_e + c_o - y_t \,\big|\, x_t\big)\big)\big(a_t^m - y_t\big), \tag{A9}$$

subject to the market clearing condition

$$G\big(b_t^m - c_e - c_o - y_t \,\big|\, x_t\big) = 1 - G\big(a_t^m + c_e + c_o - y_t \,\big|\, x_t\big). \tag{A10}$$

To solve the problem, we solve the monopolist's problem without the market clearing constraint and show that the unconstrained problem satisfies the constraint. The first-order condition for the bid quote b_t^{m*} in the unconstrained problem is

$$g\left(b_t^{m*} - c_e - c_o - y_t \,\middle|\, x_t\right)\left(y_t - b_t^{m*}\right) - G\left(b_t^{m*} - c_e - c_o - y_t \,\middle|\, x_t\right) = 0, \quad \text{(A11)}$$

which can be rewritten as

$$b^{m*} = y_t - \frac{G\left(b_t^{m*} - c_e - c_o - y_t \,\middle|\, x_t\right)}{g\left(b_t^{m*} - c_e - c_o - y_t \,\middle|\, x_t\right)}, \quad \text{(A12)}$$

and the first-order condition for the ask quote can be rewritten as

$$a^{m*} = y_t + \frac{1 - G\left(a_t^{m*} + c_e + c_o - y_t \,\middle|\, x_t\right)}{g\left(a_t^{m*} + c_e + c_o - y_t \,\middle|\, x_t\right)}. \quad \text{(A13)}$$

By the symmetry of the distribution and the zero median, the allocation at these solutions satisfies market clearing. The second-order condition for the bid quote is

$$g'\left(b_t^{m*} - c_e - c_o - y_t \,\middle|\, x_t\right)\left(y_t - b_t^{m*}\right) - 2g\left(b_{t,0}^{m*} - c_e - c_o - y_t \,\middle|\, x_t\right) \leq 0, \quad \text{(A14)}$$

and the second-order condition for the ask quote is

$$g'\left(a_t^{m*} + c_e + c_o - y_t \,\middle|\, x_t\right)\left(a_t^{m*} - y_t\right) + 2g\left(a_{t,0}^{m*} + c_e + c_o - y_t \,\middle|\, x_t\right) \leq 0. \quad \text{(A15)}$$

In order to show that the ask must be higher than the bid at the optimum, substitute the constraint into the objective function to get

$$\max_{\{b_t^m, a_t^m\}} G\left(b_t^m - c_e - c_o - y_t \,\middle|\, x_t\right)\left(a_t^m - b_t^m\right), \quad \text{(A16)}$$

which is only positive if $a_t^m > b_t^m$. The expression for the monopoly gains follows from substituting the monopolist's bid and ask quotes into the traders' optimal order submissions and taking conditional expectations. Q.E.D.

Appendix B: The Competing Risks Model

Here, we briefly describe the competing risks model. Our discussion is based on Kalbfleisch and Prentice (2002, Chapter 8) and Lancaster (1990, Chapter 5). Suppose that there are M possible failure types for a system and let T_m for $m = 1, 2, \ldots, M$ denote the associated random failure times. We observe the time at which the failure occurs, T, defined to be equal to the minimum of the random failure times,

$$T = \min(T_1, T_2, \ldots, T_M), \quad \text{(B1)}$$

and the failure type. Here, we refer to T_m as the m^{th} latent time.

In our application, there are two reasons for an order to leave the limit-order book, namely, a cancellation or an execution so $M = 2$. We observe the time at which the order leaves the book and how the order leaves the book, either through cancellation or through execution.

The hazard rate for T_m is defined as

$$h_m(\tau; Q) = \lim_{\Delta\tau \downarrow 0} \frac{\Pr(T_m \in [\tau, \tau + \Delta\tau) \mid T \geq \tau, z_t)}{\Delta\tau}, \tag{B2}$$

where Q is a vector of conditioning variables. For an independent competing risks model in which the T_ms are independent,

$$h_m(\tau; Q) = \frac{f_m(\tau \mid Q)}{1 - F_m(\tau \mid Q)}, \tag{B3}$$

where $F_m(\tau \mid Q)$ is the distribution of the m^{th} latent time and $f_m(\tau \mid Q)$ is the density of the m^{th} latent time.

Irrespective of whether the model is an independent competing risks model or not, the hazard rate for T is

$$h(\tau; Q) = \sum_{m=1}^{M} h_m(\tau; Q). \tag{B4}$$

The distribution of T is

$$F(\tau \mid Q) = 1 - e^{-\int_0^\tau h(s;Q)ds}, \tag{B5}$$

with density

$$f(\tau \mid Q) = h(\tau; Q)e^{-\int_0^\tau h(s;Q)ds}. \tag{B6}$$

The density of the m^{th} time is

$$f_m(\tau \mid Q) = h_m(\tau; Q)e^{-\int_0^\tau h(s;Q)ds}. \tag{B7}$$

Suppose we have a sample $[\tau_j, \delta_{j,1}, \delta_{j,2}, \ldots, \delta_{j,M}, Q_j]$ for $j = 1, \ldots, J$, where τ_j is the time of the j^{th} failure, $\delta_{j,m}$ is an indicator for the j^{th} failure type, and Q_j is the conditioning information associated with the j^{th} failure. Using the densities of the times, the likelihood function associated with these data is

$$\prod_{j=1}^{J} \left\{ \prod_{m=1}^{M} h_m(\tau_j; Q_j)^{\delta_{t_j,m}} e^{-\int_0^{\tau_j} h(s;Q_j)ds} \right\}. \tag{B8}$$

Taking logarithms and using the definition of the hazard rate for T, the log-likelihood function is

$$\begin{aligned}
&\sum_{j=1}^{J} \left\{ \sum_{m=1}^{M} (\delta_{j,m} \ln h_m(\tau_j; Q_j)) - \int_0^{\tau_j} h(s; Q_j)\, ds \right\} \\
&= \sum_{j=1}^{J} \left\{ \sum_{m=1}^{M} (\delta_{j,m} \ln h_m(\tau_j; Q_j)) - \int_0^{\tau_j} \sum_{m=1}^{M} h_m(s; Q_j)\, ds \right\} \\
&= \sum_{m=1}^{M} \left\{ \sum_{j=1}^{J} (\delta_{j,m} \ln h_m(\tau_j; Q_j)) - \int_0^{\tau_j} h_m(s; Q_j)\, ds \right\}.
\end{aligned} \tag{B9}$$

In estimating the parameters of distributions of the time to cancellation and the time to execution, we assume that execution times and cancellation times follow independent Weibull distributions, with hazard rates as in equation (47) in the text. Parameter estimates are found by maximizing the log-likelihood function.

Appendix C: Computing Execution Probabilities

Suppose a buy limit order at price $p_{t,b}^{buy}$ is submitted at time t. Assume that the latent cancellation and execution times have the distributions given by equations (46) and (48) in the text. The execution probability is

$$\begin{aligned}\psi_b^{buy}(z_t) &= E\left[I_t(\tau_{execute} \le \tau_{cancel}) \,\middle|\, z_t, d_{t,b}^{buy} = 1\right] \\ &= \int_0^T \left(1 - F_{cancel}\left(\tau \,\middle|\, z_t, d_{t,b}^{buy} = 1\right)\right) dF_{execute}\left(\tau \,\middle|\, z_t, d_{t,b}^{buy} = 1\right) \\ &= \int_0^T \exp\left(-\exp\left(z_t'\gamma_b^{buy}\right)\tau^{\alpha_b^{buy}}\right) \exp\left(z_t'\kappa_b^{buy}\right)\beta_b^{buy}\tau^{\beta_b^{buy}-1} \\ &\quad \times \exp\left(-\exp\left(z_t'\kappa_b^{buy}\right)\tau^{\beta_b^{buy}-1}\right) d\tau. \end{aligned} \tag{C1}$$

The second line follows from the independence of the latent cancellation and execution times, and the assumption that the latent cancellation time is bounded by $t + T$ with probability one. We compute equation (C1) numerically with T equal to two trading days, or 48,600 seconds.

Appendix D: The Conditional Likelihood Function

We observe the time and type of each order submission and the state vector at the time of each order submission. Let t_i denote the time of the i^{th} order submission, I the total number of order submissions, $d_{t_i,s}^{sell}$ and $d_{t_i,b}^{buy}$ the decision indicators, and z_{t_i} the state vector. We use these data to form a competing risks model for the order submission times. Conditioning on the common value, order quantity, x_t and z_t, using the traders' decision rule and the trader arrival rate, the hazard rate for a sell market order is

$$G\left(\theta_{0,1}^{sell}(z_t) - y_t \,\middle|\, x_t\right)\lambda(t; x_t), \tag{D1}$$

the hazard rate for a sell limit order is

$$\left[G\left(\theta_{marginal}^{sell}(z_t) - y_t \,\middle|\, x_t\right) - G\left(\theta_{0,1}^{sell}(z_t) - y_t \,\middle|\, x_t\right)\right]\lambda(t; x_t), \tag{D2}$$

the hazard rate for a buy market order is

$$\left[1 - G\left(\theta_{0,1}^{buy}(z_t) - y_t \,\middle|\, x_t\right)\right]\lambda(t; x_t), \tag{D3}$$

and the hazard rate for a buy limit order is

$$\left[G\left(\theta_{0,1}^{buy}(z_t) - y_t \,\middle|\, x_t\right) - G\left(\theta_{marginal}^{buy}(z_t) - y_t \,\middle|\, x_t\right)\right]\lambda(t; x_t). \tag{D4}$$

The hazard rate for an order submission of any type is the sum of the hazards for each different type of order submission in equations (D1)–(D4), and equals

$$\left[1 - G\left(\theta^{buy}_{marginal}(z_t) - y_t \mid x_t\right) + G\left(\theta^{sell}_{marginal}(z_t) - y_t \mid x_t\right)\right]\lambda(t; x_t). \tag{D5}$$

Substituting the hazard rates in equations (D1)–(D5) into the log-likelihood function for a competing risks model in equation (B9), the conditional log-likelihood for the order submission times is:

$$\begin{aligned}\sum_{i=1}^{I} \Bigg\{ & d^{sell}_{t_i,0} \ln\left(G\left(\theta^{sell}_{0,1}(z_{t_i}) - y_{t_i} \mid x_{t_i}\right)\lambda(t_i; x_{t_i})\right) \\ & + \left(\sum_{s=1}^{S(z_t)} d^{sell}_{t_i,s}\right) \ln\left(\left[G\left(\theta^{sell}_{marginal}(z_{t_i}) - y_{t_i} \mid x_{t_i}\right) - G\left(\theta^{sell}_{0,1}(z_{t_i}) - y_{t_i} \mid x_{t_i}\right)\right]\lambda(t_i; x_{t_i})\right) \\ & + d^{buy}_{t_i,0} \ln\left(1 - G\left(\theta^{buy}_{0,1}(z_{t_i}) - y_{t_i} \mid x_{t_i}\right)\lambda(t_i; x_{t_i})\right) \\ & + \left(\sum_{b=1}^{B(z_t)} d^{buy}_{t_i,b}\right) \ln\left(\left[G\left(\theta^{buy}_{0,1}(z_{t_i}) - y_{t_i} \mid x_{t_i}\right) - G\left(\theta^{buy}_{marginal}(z_{t_i}) - y_{t_i} \mid x_{t_i}\right)\right]\lambda(t_i; x_{t_i})\right) \\ & - \int_{t_{i-1}}^{t_i} \left[1 - G\left(\theta^{buy}_{marginal}(z_t) - y_t \mid x_t\right) + G\left(\theta^{sell}_{marginal}(z_t) - y_t \mid x_t\right)\right]\lambda(t; x_t)\, dt \Bigg\}.\end{aligned} \tag{D6}$$

In our estimation, we assume that the common value, y_t, and state vector, z_t, only change when an order is submitted. Using this assumption, we have

$$\begin{aligned}&\int_{t_{i-1}}^{t_i} \left[1 - G\left(\theta^{buy}_{marginal}(z_t) - y_t \mid x_t\right) + G\left(\theta^{sell}_{marginal}(z_t) - y_t \mid x_t\right)\right]\lambda(t; x_t)\, dt \\ &\quad = \left[1 - G\left(\theta^{buy}_{marginal}(z_{t_i}) - y_{t_i} \mid x_{t_i}\right) + G\left(\theta^{sell}_{marginal}(z_{t_i}) - y_{t_i} \mid x_{t_i}\right)\right] \int_{t_{i-1}}^{t_i} \lambda(t; x_{t_i})\, dt.\end{aligned} \tag{D7}$$

Appendix E: Formulas for the Gains to Trade

Suppose that the random variable e is distributed as a mixture of normals, with weight ρ_1 on a normal distribution with mean zero and standard deviation σ_1 and weight $1 - \rho_1$ on a normal distribution with mean zero and standard deviation σ_2. Let $a \le b$ be two constants, and $I(a \le e \le b)$ be an indicator function. Then,

$$\begin{aligned}E[I(a \le e \le b)e] = {} & \rho_1\left(\sigma_1\phi\left(\frac{b}{\sigma_1}\right) - \sigma_1\phi\left(\frac{a}{\sigma_1}\right)\right) \\ & + (1-\rho_1)\left(\sigma_2\phi\left(\frac{b}{\sigma_2}\right) - \sigma_2\phi\left(\frac{a}{\sigma_2}\right)\right),\end{aligned} \tag{E1}$$

and

$$E[I(a \leq e \leq b)] = \rho_1\left(\Phi\left(\frac{b}{\sigma_1}\right) - \Phi\left(\frac{a}{\sigma_1}\right)\right) + (1-\rho_1)\left(\Phi\left(\frac{b}{\sigma_2}\right) - \Phi\left(\frac{a}{\sigma_2}\right)\right), \tag{E2}$$

where ϕ is the standard normal density function and Φ is the standard normal cumulative distribution function.

The private value is distributed as a mixture of normals, with standard deviations $\sigma_1(x_t) = y_t\sigma_1\exp(x_t'\Gamma)$ and $\sigma_2(x_t) = y_t\sigma_2\exp(x_t'\Gamma)$. The formula for the maximum gains from trade in equation (55) in the text results from applying equations (E1) and (E2) to equations (32) and (33), using the parameterization that the order execution costs and the order submission costs are both proportional to the common value: Specifically, order execution cost $= y_t c_e$ and order submission cost $= y_t c_o$.

In order to compute the current gains from trade and the losses from no execution, no submission, and extramarginal submissions, we need to compute expectations over different intervals defined by combinations of threshold valuations and the cut-offs that define the optimal allocation.

Substituting the optimal order submission strategy for the sell side in equations (22)–(25) and the corresponding equations for the buy side into equation (27), the current gains are

$$\begin{aligned}
&\text{Current gains}\,(z_t)\\
&\quad = E\big[I\big(\infty \leq y_t + u_t \leq \theta^{sell}_{0,1}(z_t)\big)(-u_t - c_e - c_o)\,\big|\,z_t\big]\\
&\qquad + \sum_{i=1}^{S(z_t)-1} E\big[I\big(\theta^{sell}_{i-1,i}(z_t) \leq y_t + u_t \leq \theta^{sell}_{i,i+1}(z_t)\big)\big(\psi^{sell}_i(z_t)(-u_t - c_e) - c_o\big)\,\big|\,z_t\big]\\
&\qquad + E\big[I\big(\theta^{sell}_{S(z_t)-1,S(z_t)}(z_t) \leq y_t + u_t \leq \theta^{sell}_{marginal}(z_t)\big)\big(\psi^{sell}_{S(z_t)}(z_t)(-u_t - c_e) - c_o\big)\,\big|\,z_t\big]\\
&\qquad + E\big[I\big(\theta^{buy}_{0,1}(z_t) \leq y_t + u_t \leq \infty\big)(u_t - c_e - c_o)\,\big|\,z_t\big]\\
&\qquad + \sum_{i=1}^{B(z_t)-1} E\big[I\big(\theta^{buy}_{i,i+1}(z_t) \leq y_t + u_t \leq \theta^{buy}_{i-1,i}(z_t)\big)\big(\psi^{buy}_i(z_t)(u_t - c_e) - c_o\big)\,\big|\,z_t\big]\\
&\qquad + E\big[I\big(\theta^{buy}_{B(z_t)-1,B(z_t)}(z_t) \leq y_t + u_t \leq \theta^{buy}_{marginal}\big)\big(\psi^{buy}_{B(z_t)}(z_t)(u_t - c_e) - c_o\big)\,\big|\,z_t\big].
\end{aligned} \tag{E3}$$

The lower bound for the current gains from trade is obtained by using the execution probabilities for the marginal sell limit orders for all the sell limit order execution probabilities and the execution probabilities for the marginal buy limit order for all buy limit order execution probabilities:

$$\begin{aligned}\text{Current gains}\,(z_t) \geq{}& E\big[I\big(\infty \leq y_t + u_t \leq \theta^{sell}_{0,1}(z_t)\big)(-u_t - c_e - c_o)\,\big|\,z_t\big]\\&+E\big[I\big(\theta^{sell}_{0,1}(z_t) \leq y_t + u_t \leq \theta^{sell}_{marginal}(z_t)\big)\big(\psi^{sell}_{S(z_t)}(z_t)(-u_t - c_e) - c_o\big)\,\big|\,z_t\big]\\&+E\big[I\big(\theta^{buy}_{0,1}(z_t) \leq y_t + u_t \leq \infty\big)(u_t - c_e - c_o)\,\big|\,z_t\big]\\&+E\big[I\big(\theta^{buy}_{marginal} \leq y_t + u_t \leq \theta^{buy}_{0,1}(z_t)\big)\big(\psi^{buy}_{B(z_t)}(z_t)(u_t - c_e) - c_o\big)\,\big|\,z_t\big].\end{aligned} \tag{E4}$$

The upper bound is obtained by using the execution probabilities for one-tick sell and buy limit orders for all sell and buy limit order execution probabilities in equation (E3). The average gains from trade are obtained by using the average execution probabilities for the one-tick buy and marginal buy limit orders for all buy limit order execution probabilities in equation (E3). The sell limit order executions are defined similarly.

Using equations (E1) and (E2), we compute a closed form for each term in equation (E4). For example,

$$\begin{aligned}&E\big[I\big(\theta^{sell}_{0,1}(z_t) \leq y_t + u_t \leq \theta^{sell}_{marginal}(z_t)\big)\big(\psi^{sell}_{S(z_t)}(z_t)(-u_t - c_e) - c_o\big)\,\big|\,z_t\big]\\&= y_t\psi^{sell}_{S(z_t)}(z_t)\left\{-\rho\left(\sigma_1\exp(x_t'\Gamma)\phi\left(\frac{\theta^{sell}_{marginal}(z_t) - y_t}{y_t\sigma_1\exp(x_t'\Gamma)}\right)\right.\right.\\&\quad\left.-\sigma_1\exp(x_t'\Gamma)\phi\left(\frac{\theta^{sell}_{0,1}(z_t) - y_t}{y_t\sigma_1\exp(x_t'\Gamma)}\right)\right)\\&\quad-(1-\rho)\left(\sigma_2\exp(x_t'\Gamma)\phi\left(\frac{\theta^{sell}_{marginal}(z_t) - y_t}{y_t\sigma_2\exp(x_t'\Gamma)}\right) - \sigma_2\exp(x_t'\Gamma)\phi\left(\frac{\theta^{sell}_{0,1}(z_t) - y_t}{y_t\sigma_2\exp(x_t'\Gamma)}\right)\right)\\&\quad-c_e\left(\rho\left(\Phi\left(\frac{\theta^{sell}_{marginal}(z_t) - y_t}{y_t\sigma_1\exp(x_t'\Gamma)}\right) - \Phi\left(\frac{\theta^{sell}_{0,1}(z_t) - y_t}{y_t\sigma_1\exp(x_t'\Gamma)}\right)\right)\right.\\&\quad\left.\left.+(1-\rho)\left(\Phi\left(\frac{\theta^{sell}_{marginal}(z_t) - y_t}{y_t\sigma_2\exp(x_t'\Gamma)}\right) - \Phi\left(\frac{\theta^{sell}_{0,1}(z_t) - y_t}{y_t\sigma_2\exp(x_t'\Gamma)}\right)\right)\right)\right\}\\&\quad-y_tc_o\left\{\rho\left(\Phi\left(\frac{\theta^{sell}_{marginal}(z_t) - y_t}{y_t\sigma_1\exp(x_t'\Gamma)}\right) - \Phi\left(\frac{\theta^{sell}_{0,1}(z_t) - y_t}{y_t\sigma_1\exp(x_t'\Gamma)}\right)\right)\right.\\&\quad\left.+(1-\rho)\left(\Phi\left(\frac{\theta^{sell}_{marginal}(z_t) - y_t}{y_t\sigma_2\exp(x_t'\Gamma)}\right) - \Phi\left(\frac{\theta^{sell}_{0,1}(z_t) - y_t}{y_t\sigma_2\exp(x_t'\Gamma)}\right)\right)\right\}.\end{aligned} \tag{E5}$$

The other terms and the upper bound are computed similarly.

The no execution losses, no submission losses, wrong direction losses, extra-marginal submission losses, and the monopoly gains are computed similarly.

Appendix F: Reweighting the Sample

Using the definition of conditional probability,

$$\Pr(z_t \mid \text{Trader arrives})\Pr(\text{Trader submits an order} \mid z_t, \text{Trader arrives})$$
$$= \Pr(z_t, \text{Trader submits an order} \mid \text{Trader arrives}), \quad \text{(F1)}$$

or

$$\Pr(z_t \mid \text{Trader arrives}) = \frac{\Pr(z_t, \text{Trader submits an order} \mid \text{Trader arrives})}{\Pr(\text{Trader submits an order} \mid z_t, \text{Trader arrives})}. \quad \text{(F2)}$$

Also from the definition of conditional probability,

$$\Pr(z_t, \text{Trader submits an order} \mid \text{Trader arrives})$$
$$= \Pr(z_t \mid \text{Trader arrives and submits an order})$$
$$\times \Pr(\text{Trader submits an order} \mid \text{Trader arrives}). \quad \text{(F3)}$$

Substituting equation (F3) into equation (F2) and rearranging gives equations (59) and (60) in the text.

We use our first-stage and maximum likelihood estimators to estimate $\omega(z_{t_i})$, which from equation (60) depends on

$$\Pr(\text{Trader submits an order} \mid \text{Trader arrives})$$

and

$$\Pr\big(\text{Trader submits an order} \mid z_{t_i}, \text{Trader arrives}\big).$$

From the theoretical model,

$$\Pr(\text{Trader submits an order} \mid z_t, \text{Trader arrives})$$
$$= G\big(\theta^{sell}_{marginal}(z_t) - y_t \,\big|\, x_t\big) + 1 - G\big(\theta^{buy}_{marginal}(z_t) - y_t \,\big|\, x_t\big). \quad \text{(F4)}$$

We use equation (F4) evaluated at the first-stage and maximum likelihood estimates to form

$$\widehat{\Pr}\big(\text{Trader submits an order} \mid z_{t_i}, \text{Trader arrives}\big). \quad \text{(F5)}$$

Let $\mathcal{Z}$ be the the state space for z_t. Integrating over $\mathcal{Z}$ and using equations (59) and (60) in the text,

$$1 = \int_{\mathcal{Z}} \Pr(z_t \mid \text{Trader arrives})\, dz_t$$
$$= \int_{\mathcal{Z}} \Pr(z_t \mid \text{Trader arrives and submits and order}) \times \omega(z_t)\, dz_{t_i}.$$
$$= \Pr(\text{Trader submits an order} \mid \text{Trader arrives})$$
$$\times \left(\int_{\mathcal{Z}} \frac{\Pr(z_t \mid \text{Trader arrives and submits and order})}{\Pr(\text{Trader submits an order} \mid z_t, \text{Trader arrives})}\, dz_t\right), \quad \text{(F6)}$$

where the last line follows from the definition of $\omega(z_t)$ in equation (60) and the fact that

$$\Pr(\text{Trader submits an order} \mid \text{Trader arrives})$$

does not depend on z_t. Equation (F6) implies that

$$\begin{aligned}&\Pr(\text{Trader submits an order} \mid \text{Trader arrives})\\&= \left(\int_{\mathcal{Z}} \frac{\Pr(z_t \mid \text{Trader arrives and submits and order})}{\Pr(\text{Trader submits an order} \mid z_t, \text{Trader arrives})} dz_t \right)^{-1}. \quad \text{(F7)}\end{aligned}$$

Substituting in the empirical distribution for $\Pr(z_t \mid \text{Trader arrives and submits an order})$ in the numerator and our estimate in equation (F5) in the denominator, we form

$$\begin{aligned}&\widehat{\Pr}(\text{Trader submits an order} \mid \text{Trader arrives})\\&= \left(\sum_{i=1}^{I} \left(\frac{1}{I}\right) \left(\frac{1}{\widehat{\Pr}\left(\text{Trader submits an order} \mid z_{t_i}, \text{Trader arrives}\right)} \right) \right)^{-1}. \quad \text{(F8)}\end{aligned}$$

Using equations (F5) and (F8), we form the consistent estimate

$$\hat{\omega}(z_{t_i}) = \frac{\widehat{\Pr}(\text{Trader submits an order} \mid \text{Trader arrives})}{\widehat{\Pr}\left(\text{Trader submits an order} \mid z_{t_i}, \text{Trader arrives}\right)}. \quad \text{(F9)}$$

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