1 Introduction

Markets emerge as institutions for voluntary exchange where there are many buyers and sellers of large numbers of close substitutes for a product, and a medium of exchange (such as fiat money or gold) or ledger balances to record debits and credits. The close substitutes are typically viewed by market participants as equivalent goods, or different units of the same product, but that need not literally be true. Because the various units for sale are so similar to each other, the contractual arrangements associated with a market transaction (such as a warranty) are relatively homogeneous. Consequently the differences between the units known at the time of the exchange are not important enough for either the buyer or the seller to bargain about. This feature distinguishes markets from bargaining and contracting, which are often tailored to a uniquely defined situation (such as house repair) that takes time and energy to assess.This chapter extends our earlier analyses of bargaining, auction and monopoly games to market settings.

demand and supply

The first parts of the chapter investigate the extent to which monopolistic practice breaks down with competition from rivals. Section 2 explores how market power on one side is dissipated through competition by rivals. Starting with duopoly, models where two firms supply many demanders, we derive the solution price and quantities produced by both firms for various cost structures and strategy spaces. Both affect the solution outcome. For example if firms compete on price rather than quantity the outcomes might differ markedly, depending on whether there are capacity costs or not. Building upon our analysis of duopoly, we derive the solution for an arbitrary numbers of firms and analyze the limiting behavior of the industry as the number of supplier firms increases.

Nonconstant costs

Can we endogenously determine the number of entrants? We study how the number of sellers is determined within the equilibrium as a mapping from the underlying technology. We argue that there are essentially two cases to examine, constant costs of production and declining costs of production. An important special case of the latter involves setup costs, and this is the one we focused upon. If there are fixed entry costs, how many will enter? Can we predict entry as demand expands? How about contraction as demand shrinks? This leads to a discussion of entry deterrence. When the number of firms is endogenous there is scope for early entrants to deter potential entrants from competing in the industry. This can occur is several ways. Capacity expansion when setup costs are endogenous. Shielding the firm from information about the other firms so that response is slow to new competition, which would drive down the price of the product for several periods before a more cooperative could be reached. Third is incomplete information about the cost structure of the pioneering firm, and/or about the demand for the product itself.

Whereas the identitiy of players as buyers and sellers are exogenously determined in Sections 2 and 3, the latter sections of this chapter turn to market institutions where the direction of trade is endogenously determined. Trading exchanges. Limit order market games are suitable for analyzing many different types of trading mechanisms, including, but not limited to, retail markets, as well as most of the auction and bargaining games we discussed in the previous chapters. Section 4 lays out the components of limit order markets.

Then in Section 5 we study some examples of limit order markets, first an equilibrium in simple markets for a small number of traders, where traders with private valuations and endowed with asset units, then markets that are mediated by dealers or specialists. We show that when there are an odd number of traders, there is an efficient decentralized trading mechanism in which all trading takes place at the median valuation. When there is an even number all but the two median traders exhaust the potentail gains from trade. Regardless of whether the number of traders is odd or even, and irrespective of the size of each trader, as the number of traders in a simple trading market increases, and the size of each trader diminishes relative to the market potential, the solution to this game converges to the goods allocation and price attained in the corresponding competitive equilibrium, defined in the final chapter.

The general solution to limit order market games is intractable. Nevertheless there are certain properties that all game theoretic solution have, and these are discussed in Section 6. Briefly, trading should not occur because of differences in information alone. There should be not arbitrage possibilities in the solution. No asset returns should exhibit first order stochastic dominance in the solution, and if traders are risk avoiders, second order staochastic dominance either. Finally traders with more extreme valuations for assets are more likely to place orders that are executed.

2 Oligopoly

This section investigates an industry where there are a fixed number of sellers competing against each other to supply a homogeneous product to a consumer population. We start by analyzing the case of two producers, duopoly. Then we extend our analysis to cover any finite number of suppliers, and show how the industry is affected as that number diverges.

2.1 Price setting duopoly with constant costs

The effects of competitive rivalry are most ruinous when firms compete on price. Suppose two retail stores labelled $i \in \{1,2\}$ supply a market of consumers $j \in \{1,\ldots,J\}$ with an identical good x. Each store simultaneously sets its own price for the good, denoted p_1 and p_2 respectively. Once both prices are announced, consumers buy as many or as few units of the good as they choose from their preferred store. Producers pay the same unit cost c to their wholesalers, which is independent of their sales. We show below that the unique solution to this game is for both firms to price at cost. The derivation of the solution does not rely on any special assumptions about consumer valuations for the good, or what retailers know about those valuations. We denote that valuation of the j^{th} consumer for x_j units by $v_j(x_j)$, and merely assume that there is strictly positive demand for x at some price above c by at least one consumer. That is $v_j(x_j) > c$ for some $x_j > 0$ and $j \in \{1,\ldots,J\}$.

Since consumers collectively make the last move in the game with their individual purchases, each consumer buys at price $\min\{p_1,p_2\}$ from a store charging the lowest price. Consequently if one store priced below cost, then at least one of the stores would make a loss. Therefore charging less than cost is weakly dominated by setting $p_i \geq c$ for $i \in \{1,2\}$. Now suppose the first store prices the product strictly above cost. Then the best reply of the second store is to set p_2 between c and p_1. In that event the first store would make no profit, but the second firm would. By symmetry this is not a best response for the first firm. Therefore neither firm prices strictly above cost in Nash equilibrium. Upon checking that $p_1 = p_2 = c$ is a solution to this price setting game in which the stores make neither profits or losses, it must be unique.

The arguments and the solution given above can be modified to deal with different supply costs. Suppose the costs for the first and second stores are c_1 and c_2 respectively, with $c_1 < c_2$. For the reasons we gave above, charging less than cost is weakly dominated by setting $p_1 \geq c_1$ and $p_2 \geq c_2$. Moreover price competition implies $\max\{p_1,p_2\} \leq c_2$. Therefore $p_2 = c_2$ in equilibrium. We now show that in every solution to this game the low cost strore captures all the market. There are two cases to consider. If $p_1 < c_2 = p_2$, then all consumers will frequent the first store, as claimed. Now suppose $p_1 = p_2 = c_2$ and some consumers purchase from the second store. This does not solve the game because the best response of the first firm is to marginally lower its price and capture all the market. The upshot is that $p_2 = c_2$ in every solution to the game, and the first firm chooses $p_1 \in [c_1,c_2]$ to maximize profits from supplying the whole market. Let p_m denote the solution to the constrainted monopoly problem. If $p_m < c_2$, then the presense of a rival store is immaterial, because this is what the first store would charge if it held a monopoly. If $p_m = c_2$, then both stores charge c_2 and all the customers frequent the first store.

These results extend to industries where there are more than two stores sereving the market. As before we subscript firms by $i \in \{1,\ldots,I\}$ where $I \geq 2$, and rank them by their supply costs, meaning $c_1 \leq \ldots \leq c_I$. If several stores all share the same cost as the first firm, then demanders buy from them at price c_1, and the analysis of the equal cost price setting duopoly case applies with minimal revision. Alternatively if $c_1 < c_2$ the analysis of the previous paragraph applies in its entirity.

Experiment 19.1

Suppose the valuations of demanders are private, drawn from independently from a uniform distribution. Does the outcome of the game depnde on the number of suppliers beyond two? Suppose there are only two firms but they have different costs structures.

2.2 Matching Prices

The preceding discussion on price setting duopolies suggests that even a small number of firms can drive the solution price close to cost dissipating monopoly rent. For this reason we might expect stores to exapnd their strategy spaces so that a more profitable outcome can be achieved. For example, some stores have a policy of matching prices on identical products available at rival stores. We now consider a price setting game in which all stores in the market can guarantee the lowest price.

Suppose there are I stores, and the unit procurement and selling cost for each store is identical at c. At the beginning of the game the stores simultaneously post a price for the good. As before we denote by p_i the price posted by store $i \in \{1,\ldots,I\}$. Consumers then review the prices $\{p_1,\ldots,p_I\}$ and visit any store to purchase as many units as they wish at any of the listed prices, that is regardless of whether the strore they are visiting had posted that price or not.

In the solution to this game all consumers purchase at the lowest price offered, defined by

$$p_{\min} = \min\{p_1,\ldots,p_I\}$$

Anticipating this market response, each store recognizes the price it posts will not affect its transactions unless it is the lowest, and that it cannot affect its demand through price. Denoting by π_n the fraction of customers who purchase the good from the n^{th} store, and $Q(p)$ teh expected total sales from all stores when price p is announced, the n^{th} store chooses $p_{\min}$ to maximize

$$\pi_n(p_{\min} - c)Q(p_{\min})$$

The solution to this optimization problem has two remarkable features. First, regardless of the market share served by each store, they all choose the same price. Second, the common price they pick is p_m the monopoly price (verified by setting $\pi_n = 1$ in the expression for profits).

Guaranteed minimum pricing effectively prevents one store from undercutting another to scoop out demanders. In contrast to games where price competition drives producers back to marginal costs the collective behavior of consumers unwittingly help producers to restrict ouptut and realize all the monopoly rent by keeping each producer aware of his competitor’s prices. From the perspective of consumers this is another application of the prisoner’s dilemma we discussed in Chapter 6 on dominant strategies. If consumers could conspire to prevent stores from knowing the amount their competitors discounted, then their rivalry would generate marginal cost pricing.

Experiment 19.2

There are several ways of playing games where producers are forced to match prices. One mechanism is for producers to submit limit orders, customers to submit market orders, and for firms to automatically submit refunds, or retrospective discounts, on all items purchased at prices above the minimum limit sell order submitted to customers by either firm. Compare the outcome with the monopoly solution.

2.3 Quantity setting duopoly with constant costs

An alternative to competing on price is to compete on quantity, by choosing production levels and let demanders determine a uniform price that maintains inventory levels. For example all the goods produced by the suppliers might be sold in a multiunit sealed bid auction where bids from demanders are ranked from the highest to the lowest, units are succesively allocated to the top bidders to clear supply, every buyer paying and every seller the unit price of the lowest successful bid.

Let $P(q)$ denote the expected price the firms expect to receive whena total of q units is offered for sale at the auction market. We suppose $P(q)$ is decreasing and differentiable in q with derivative $P'(q)$. Supposing there are J producers

$$q = \sum_{j=1}^{J} q_j$$

If unit costs are c for each firm, the k^{th} firm chooses q_k to maximize

$$P\left(\sum_{j=1}^{J} q_j\right)q_k - cq_k$$

The first order condition for an interior maximization is

$$P'\left(\sum_{j=1}^{J} q_j\right)q_k + P\left(\sum_{j=1}^{J} q_j\right) = c$$

In a symmetric equilibrium all firms produce the same quantity, meaning $Jq_j^o = q^o$ for all $j \in \{1,\ldots,J\}$ and some positive q^o. In that case teh equilibrium quantity produced by each firm can be solved from the equation

$$P'(q^o)q^o + JP(q^o) = Jc$$

The limiting properties of this expression are noteworthy. In teh case of monopoly $J = 1$ and the industry equilibrium quantity is simply the monopolist's procudtion. We obtain the standard condition that marginal revenue is equated with marginal cost. We may deduce what happens to industry outuput as the number of firms increase by first dividing the equation determining output by the number of firms, and then taking the limit as J increases wtithout bound. Dividing the first order condition by J and rearranging, we see that

$$P(q_J^o) - c = \frac{P'(q_J^o)q_J^o}{J}$$

Taking the limit we now obtain the result that the equilibrium quantity sold

$$\lim_{J\to\infty} \left\{\frac{P'(q^o)q^o}{J}\right\} = 0$$

which implies $P(q_J^o)$ converges to c as J increases.

For example, if $P(q)$ is linear in q and defined as

$$P(q) = \alpha_0 - \alpha_1 q$$

then upon substituting expressions for $P'(q^o)$ and $P(q^o)$ into the solution for q^o we obtain

$$-\alpha_1 q^o + J[\alpha_0 - \alpha_1 q^o] = Jc$$

which simplifies to

$$q^o = \frac{J}{(J+1)} \frac{(\alpha_0 - c)}{\alpha_1}$$

In the linear model with constant costs a monopolist produces half the efficient quantity of $(\alpha_0 - c)/\alpha_1$, a duopolist two thirds and so on.

Experiment 19.3

Do our experiments suggest that the limit point depends on the cost structure?

Another question is how many firms are required to reach this limit (that is when it exists).

Again we suppose firms set prices, which consumers take as given.

How are things affected?

This set of experiments explores different ways in which the competitive limit might be reached. Suppose there are K retailers selling identical products, and consumers shop between the outlets. Conduct experiments as K increases from one to five, reporting on price dispersion in the market, quantity sold, and the distribution of the valuations to consumers versus non consumers.

2.4 More on the cost structure

Our analysis has focused exclusively on firms that face constant costs in production. This is is a reasonable starting point. If the factors in production can be replicated, then ap

There are, however many reasons to suppose that the cost structure is not constant. If ingredients are mixed in constainers and the costs of the containers are sig experitse is applied cheaply on teh other hand the span of control of managers is limited, the advantages of local market knowledge, costs are partly fixed and partly variable.

How sensitive are our results on oligoploy to different assumptions about the cost structure?

Increasing costs and decreasing scale returns

Capacity commitments

Decreasing costs and increasing scale returns

3 Entry and Exit

The discussion above analyzed games where the number of firms is fixed. In reality new firms enter an industry when the prospects appear profitable, and exit if the expected present value of future revenues fail to cover the expected discounted sum of future operating costs and creditor debt. Changes in the production technology, uncertainty about demand, and unforeseen actions by rivals all affect revenue and cost flow calculations and hence entry and exit decisions. This leads us into a discussion of entry and exit by firms in response, and in anticipation of, changing product demand and input supply conditions. The results we derive in this section are based on dominance explained earlier in this chapter. There we showed that if firms set prices, the solution for the industry is for each firm to charge marginal cost, but if firms set quantities, then the solution would entail each firm making strictly positive profits. We now explore extenstions of these results.

Suppose that firms entering an industry are not subject to any sunk or fixed costs. In this case all costs are variable, there are three main scenarios to investigate. Costs increase more than proportionately with prodiction, called decreasing scale returns, costs increase proportionately with production, constant scale returns, and total costs increase less than proportionately, increasing scale returns. It is easy to see that if there are decreasing returns to scale, firms will enter and exit at the minimum feasible scale and the equilbrium price will match the marginal cost at that scale. Furthermore, assuming the marginal (variable) cost is independent of the production level, and entry is free, then the number of firms in the industry, and the output of individual firms is indeterminate, and the equilibrium price is the marginal cost. In both the dereasing scales return and constant scales return cases, producers sell units to all demanders with valuations exceeding the unit cost of production, and nothing to demanders whose valuations lie below the cost of production. In both cases industry output adjusts to preserve parity between price and the industry cost of producing a marginal unit.

Investigating the nature of equilibrium when firms have constant variable cost but, in addition, must invest resources to enter the industry, and/or incur some costs regardless of their production level gives insight into the more general case of increasing scale returns in production. In analyzing industries where there are significant natural barriers to entry, it is important to distinguish between sequential entry in perfect information games, where successive firms observe the number of incumbent firms who have already committed to enter and those who have yet to commit, and simultaneous move games, where firms make their entry decisions without observing the decision of their rivals. Our discussion treats these cases separately.

In many real life applications, the order in which potential rivals would enter the industry is common knowledge. The opportunity to enter might hinge on local knowledge that favors firms with operating units in the geographic region, markets for new products might be more quickly recognized by firms producing closely related goods in product spectrum, the production technology for a market might be related to what the firm already does, regulations may favor some some firms over others. Call them natural barriers. All of these factors favor some firsm over others, but the longer teh horizon the less critical the advantages might be. Thus in this subsection we extend some our first examples in Chapter 3 on sequential location, and sequential location games considered in Chapter 12 to models where the timing of entry and the nature of the ensuing rivalry are determined within the solution to the game.

Accordingly consider an industry with N potential entrants, which enter sequentially following an order that is endogenously determined as part of the game. We assume that the n^{th} firm is thus defined by two parameters (f_n, c_n), where f_n is a fixed entry cost, and c_n is the unit cost of production, assumed independent of scale. Demand for the service is generated by a fluctuating population of consumers numbering Q_t at time t with valuations drawn from a probability distribution denoted by $F_t(v)$. It is convenient to deal with each of these separately as subcases.

If there is price competition upon entry, none of the firms make any rent upon entry. This implies only one firm is willing to enter the industry. Therefore the first entrant will earn monopoly rents each period. Firms thus compete for the advantage of being first entrant by investing in new product lines that may never be be marketable, and introducing new products long before demand supports incurring a fixed cost in order to stake out territory, and thus deter rivals from entering the industry.

To capture these features in a game suppose the current value at time t from being a monopolist in the industry from that time forwards is denoted by V_t. This value is computed by discounting the sales and costs made at each time $s \geq t$ back to the current period t, and then summing over all future times $s \geq t$. This is the value of the continuation game for the first entrant.

We suppose that if any firm enters the industry sufficiently early, they will initially make losses, but that the later they enter teh industry, the smaller these losses are.

If the costs of entry are fixed at f, then a firm enters the first time V_t rises above f. In other words all the gains from the monopoly rent are dissipated by a competition for early entry by the first firm.

This is the most stark case of preemptive entry because all the rents from the monoplistsic power are exhausted by rivalours competition to early entry. For example in teh case of a monopolitic who makes ultimatum offers to each consumer knowing their valuations, then the net value of the new product would be zero. Technological and product innovation would not make anyone better off!

Matching Prices

When there is scope for collusion after entry occurs, or if firms within the industry compete by matching prices, then more than one firm is enticed to enter. Let us suppose that J firms enter sequentially. The last firm to enter pays f_{J+1} and in return receives $\pi_J V_t$ of teh monopoly profits. this is teh latest the J^{th} firm can delay entry to preempt entry by the firm with the next lowest costs. From that point onwards it is not in the interest of any other firm to enter. Solving by backwards induction, the $(J-1)^{th}$ firm enters tthe first time

$$\pi_{J-1}(V_t - \beta^{(J)}) + \pi_J \beta^{(J)} V^{(J)} \geq f_J$$

We solve this problem inductively usings the methods introduced in Chapter 3.

Capacity Choices

Another aspect to market rivalry

Invest heavily for high capacity, or little for low capaicty. We consider a menu of choices denoted by (f_i, c_i) where f_i is a fixed cost and c_i is a capacity, or a unit variable cost. Only those (f_i, c_i) on the technology frontier are considered as investmetn possibilites by the firm. Which firms enter and when?

Is investment capacity this tied to operating costs?

Is there a provision to shut down?

Once and for all?

Capacity to operate at on level only, open up versus shut down

Simultaneous Moves

For reasons we have mentioned, it may seem more plausible to regard firm entry and exit decsions as a sequential process. Nevertheless when new products are being developed, a firm is not necesssarily privy to what their potential rivals are doing. In order to reduce the probability that their findings will be stolen, copied, or exploited in ways that are detrimental to profits, many firms hide details about the nature and extent of their research from other firms. This also prevents rivals from channelling their own research in directions that mutually beneficial to the industry group. In this case entry into the new market is simultaneous. That is each firm chooses whether to incur expenditure necessary to enter the market without knowing the decisions of its potential rivals. The fruits of research are by definition unpredictable, but the strategic lack of coordination between firms moving simultaneously only excerbates the riskiness of creating new markets through product development.

Accordingly we now a consider a game in which N firms simultaneously decide whether to enter an industry, and then the entrants market the product. As in sequential games, the entry decision is of the firm hinges upon the continuation game that entrants play. We appeal to the assumption of symmetry in the post entry game, which in this context implies that each firm compares about the number of entrants in the continuation game, but not their identity. Let V_{nj} denote the continuation value of the game to the n^{th} firm upon entry into the market when the total number of entrants is $j,$ a positive integer between 1 and $N.$ For example in the post entry game, V_{n1} is the value of having a monopoly, $V_{nj} = 0$ if $j > 1$ and firms compete on price, $V_{nj} = \pi_j V_{n1}$ if $j > 1$, the n^{th} firm supplies a fraction π_j of the market for some $\pi_j \in [0,1]$ and firms match on price, and so on. Folding the continuation game back to obtain the reduced simultaneous move game in which each firm decides whether to enter or not.

The strategy space for each firm in the reduced simultaneous move game is represented by the indicator variable

$$d_n = \begin{cases} 1 \text{ enter} \\ 0 \text{ do not enter} \end{cases}$$

If the n^{th} firm enters, it incurs a fixed cost of $f.$ To derive a formal expression for the payoff, first note that it can only be nonzero if $d_n = 1.$ Conditional on the firm entering, the number of entrants is an integer between 1 and $N.$ Let the indicator function $1\left\{\sum_{k=1}^{N} d_k = e\right\}$ take a value of 1 if the number of entrants is e and 0 otherwise. The n^{th} firm is thus worth $(V_{nj} - f)$ if $1\left\{\sum_{k=1}^{N} d_k = j\right\} = 1$ and also $d_n{=}1,$ but nothing otherwise. Consolodating this within a formula, the payoff to the n^{th} firm for the reduced simultaneous move game is

$$d_n \sum_{j=1}^{N} \left[1\left\{\sum_{k=1}^{N} d_k = j\right\} (V_{nj} - f) \right]$$

Following our usual convention we use a supercript e to designate Nash equilibrium strategies. We accommodate the possibility of mixed strategies by defining $p_n^e \in [0,1]$ as the probability that $d_n = 1.$ The probability that $j - 1$ firms enter apart from firm n is then

$$\begin{aligned} \Pr\left[\sum_{\substack{j=1 \\ j\neq n}}^{N} d_j = j - 1 \right] &= \Pi p_j^e \Pi (1 - p_j^e) \\ &= \pi_{j-1}^e \end{aligned}$$

In equilibrium, the n^{th} firm enters if

$$\sum_{j=1}^{N} \pi_{j-1}^e (V_{nj} - f) > 0$$

and $d_n^e = 0$ otherwise.

Experiment 19.4

Compare games of sequential entry with those simultaneous entry.

4 Limit Order Markets

The previous section demonstrates how the strategy space for the game affects the price and quantity allocations in equilibirum, and provides conditions under which the solution efficiently allocates resources when players are assigned to be on one side of the market or the other. In many applications in industrial organization this is not a severe restriction. Demanders of manufactured goods, for example, do not ordinarily consider making and selling the products regarless of the prices they anticipate in the future. This assumption does not, however, fit financial markets, where those who hold securities buy and sell them at different times depending on thier price and dividend prospects.

Electronic limit order markets are amongst the fastest growing forms for trading financial assets and are thus important in their own right. Furthermore speicilziations of limit order market games also approximate the trading rules in many real commodity markets. This section desribes the structure of limit order market games. We define the objects of trade, and explain how trading is conducted in limit order markets. Traders are constrained by their budget constraints, which limit the orders they can place. We define the budget constraints facing each trader, the information players have, and the characterize their preferences.

4.1 Trading

At the beginning of the game players are endowed with a vector of commodities and a money balance with which they can trade during the game. Trading in J goods for money takes place in continuously over the time interval $[0,T]$. Let x_{jnt} denote the stock of the j^{th} commodity held by the n^{th} trader at time $t \in [0,T]$ and m_{jnt} his cash balance. For convenience we assume that the set of feasible transaction prices is countable, denoted by $P \equiv \{p_1, p_2, p_3, \dots\}$, and in this manner define a unit of monetary account. All trades exchange money for units of a given commodity, so there are J markets open at each point in time $t \in [0,T]$.

Players can trade by placing a limit order or a market order. A limit order is a proposal to trade at the terms defined by the person submitting it. This includes the quantity she is offering for sale, or requesting to buy, the price she will accept, or is willing to pay, and the length of time the order is active. For example, when a bidder submits a limit order to buy one unit at ten dollars if seller accepts the offer within two minutes, the bidder is obliged to pay ten dollars for a unit if any supplier approaches her within two minutes and wants to sell her one for ten dollars. A limit order to buy a specified number of units at a designated price legally obliges the trader submitting the order to honor his commitment if another trader seeks to sell any quantity less than or equal to the specified number at that price. Furthermore if a seller only fulfills part of the limit order, then the limit order buyer remains committed to buying the remaining units if another seller wishes to exercise his option to sell at that price. When a limit order is placed, we say it becomes active, and if its time limit is exceeded before being crossed, we say that the order expires.

Every trade on a limit order market crosses a limit order to sell with a market order to buy, or vice versa. Submitting a market order amounts to accepting the trading proposal defined by some limit order. Market orders to sell a given quantity represent agreements to accept the most attractively priced limit orders to buy. A market order to buy (sell) one unit is defined by a price which is greater (less) than or equal to the lowest (highest) outstanding limit order to sell (buy). They are executed immediately. Market buy (sell) orders

Market orders to sell (buy) are matched with the highest (lowest) priced limit order to buy (sell) and executed at the price of the matching limit buy (sell) order, reducing the number of outstanding limit orders to buy(sell). If there is more than one active limit order to buy, and a supplier places a market order to sell units of the good or service, the limit orders at the highest price are filled first. More generally, market orders garner the most generous prices from the set of active limit orders. When two active limit orders offer identical prices and quantities, the order submitted earliest is fully executed before any of the order submitted most recently is executed. For example if two limit orders to buy are submitted at the same price, the order submitted first is matched against a market sell order before the more recently submitted buy order. Limit order markets typically permit players submitting limit orders to withdraw them any time before they are matched with incoming market orders as part of a transaction.

4.2 Limit order book

The limit order book or trading window displays the active limit orders. In Figure 17.1 below, the trader in this market has just placed a sell order for 9 units at price 5,800, with an expiry time of 60,000 seconds (that is 16 hours 40 minutes). There are 6 limit orders to buy already in the books (2 at 3800 and 4 at 200), and 4 other limit orders to sell at 6,000.

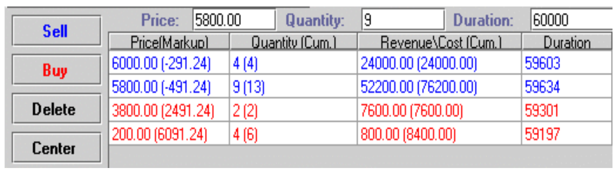

Figure 17.1
Trading window

The spread is defined as the difference between the highest priced limit buy order ask price, called the bid price, and the lowest priced limit sell order called the ask. In the example above, the ask price is 5,800 and the bid price is 3,800, so the spread is 2,000. Observe that the trader whose display screen is illustrated reduced the spread from 2,200 by placing an order inside the previous bid ask quotes. Thus the spread is also the loss from buying and then selling a share.

4.3 Solvency

At the beginning of the game the n^{th} player is endowed with a commodity allocation $(x_{jkt}, x_{jkt}, \ldots x_{jkt})$ and an initial balance m_{jkt}. The trader is constrained at each point in time by how much she can offer for sale within each market and how much she can buy in total. During the game, intertemporal budget constraints limit the trader’s choices. We assume players cannot place orders to sell shares they do not actually own, that is to sell short. Similarly all buy orders must be covered by cash at the time of submission. Since fraud and theft are not possible in such games, these two conventions fully characterize property rights. As stated before there is only one medium of exchange in limit order market games, called money: all stock is sold and bought with money. For example stock swaps and currency exchanges are excluded from limit order markets, although much of what we analyze here could be broadened to cover these situations.

The short sale constraint implies the total amount of each asset up for sale cannot exceed her holdings. Let s_{jkt} denote the quantity of the j^{th} asset for sale at price k at time t. The J short sale constraints are expressed as:

$$\sum_{k=1}^{\infty} s_{jkt} \leq x_{jkt}$$

for each $j \in \{1, \ldots, J\}$.

A bankruptcy constraint on buy orders prevents the trader from placing orders that exceed her money holdings. It effectively constrains the seller from exchanging (selling) more money for assets than she holds. Let b_{jkt} denote the quantity of the j^{th} asset demanded at price k at time t. The single bankruptcy constraint requires:

$$\sum_{j=1}^{J} \sum_{k=1}^{\infty} p_j b_{jkt} \leq m_t$$

Taken together, the initial endowment and the solvency constraints can be used to define what limit order strategies are (dynamically) feasible.

Third we may place additional constraints on certain player types,by only permitting them to make limit orders in some markets and market orders in others, and/or by restricting the direction of their orders (such as permitting them to buy but not sell).

4.4 Information

The information each player has at the beginning of the game, and what players can observe throughout the game helps determine the nature of their orders, and hence trading. Players might have different information at the start of the game, about the endowments of others, and the initial valuations. Their information about changing factor values might be might be updated with more or less precision, perhaps with a lag. Some players might have detailed information about all the factors, oher players information about only some of them. Players might be more or less informed about the contents of the order book, which keeps a record of all outstanding limit orders, and about the history of transactions. Changing the information structure in a limit order market game clearly affects both the equilibrium strategies and also the experimental outocmes observed. This last point has already made in connection with the auciton games.

For example, knowing the other players' endowments can affect a bidder's strategy. If a bidder knows that all his rivals are endowed with a limited budget for the auction, this sets an upper bound on his own bidding strategy. by bidding just a little more he can be assured of winnning. From the trader's perspective, it is likewise relevant to know details how the endowments of the other players evolve over time.

Another piece of information of interest to players is the limit order book, which in the case of auctions records their bids. The essential difference between English auctions and first price sealed bid auctions is that bidders in English auctions observe the highest buy limit order, called the best buy, whereas bidders in first price sealed bid auctions observe no new information once the auction begins. Bidders in first price sealed bid auctions cannot justify revising their initial offers. Bidders in English auctions, however, raise their initial limit buy orders, or withdraw from the auction, in response to limit order buy submissions by other bidders.

Differences in knowledge about valuations also distinguish auctions from each other. For example a common value first price sealed bid auction differs from an auction where one of the bidders knows the common value. As we have seen, the uninformed bidders play a mixed strategy in differential information auctions, but pure strategies in a common value auction.

4.5 Preferences

In the spirit of the axiomatic approach given in Part II, we assume the preferences of the n^{th} player can be order by a utility function, denoted here by $u_n(x_{nT}, m_{nT})$, which is increasing in each argument. We also assume that the players obey the expected utility hypothesis, which we now denote by

$$E[u_n(x_{nT})|I_{nt}]$$

Given a set of primitives, which for convenience we now call assets, perhaps the simplest assumption to make about preferences in market games is that traders maximize the expected sum of their value weighted assets held at the end of the game. Accordingly, suppose v_{jnt} is the unit value of an asset $j \in \{1, \ldots, J\}$ to player $n \in \{1, \ldots, N\}$ if the game were to end instantly at time $t \in [0, T]$. Also let x_{jnt} and m_{nt} respectively denote the quantity of the j^{th} asset and the amount of cash held by trader n at time t.

Finally there is an important distinction to be made between trading games we have described and analyzed where money is an asset with inherent value, distinguished only by its special role as being the sole medium of exchange., versus games in which money has no value apart from as a medium of exchange, and as a tangible measure for counting liabilities and credits. We define fiat money money by its two distinguishing features: it is a fixed stock of divisable resource with no inherent value that can be easily transported and exchanged. In many trading games it is innocuous to assume that money stands for generalized purchasing power, backed by real assets that must be surrendered to cash holders on demand. Nevertheless examining when and why fiat money is used gives insight into the strategic narture of contractual obligations and conventions. In a game with a fixed finite horizon the backwards induction arguments we have used to establish the nature of equilibrium in perfect information games provide an answer to this question with devasting clarity. No trade takes place in a finite lived economy if traders are restricted to exchanging money for goods. The fact that currency is widely used suggests that there are other factors to explain its existence. One notion is that an economy is never ending, and that using fiat money might be acceptable since younger generations accepting cash anticipate that they will be able to continue using . A similar argument has made to justify the funding of social security schemes in growing economies. The evidence that monetary systems break down when there is a crisis of confidence in a regime's ability to sustain production and trade also suggests that arguments for the accpectance of fiat money depend on the continuing survival of the economy. Both propositions, that fiat money will not be used in finite lived economy, but might be adopted in a economy that is long lived, can be investigated within an experimental setting.

Exercise *Here we consider the*

1. *There are J assets and K factors. The trader holds. Compute the mean return and its variance.*

2. *In a trading game for N players, each trader $n \in \{1,\ldots,N\}$ holds a nontradeable asset whose return is fully determined by a single factor denoted f_n. In addition there are several other tradeable assets, whose returns are different linear combinations of $f = (f_1,\ldots,f_N)$. The mean return of each asset is the same. Conduct a trading game experiment and investigate the following.*

Exercise *We conisder three economies in which there is fiat money, and three where the maony is backed by a real asset.*

1. *An economy where there is a finite ending time*

2. *An overlapping generations economy*

3. *A circle economy*

5 Examples

Many auctions are specializations of limit order markets. In first price sealed bid auctions bidders submit limit orders to buy the item and the auctioneer submits a market order to end the game. The fact that the bids are sealed corresponds to hiding the entries in the limit order book from the bidders at the time they submit their orders. Similarly, in an oral auction bidders submit only limit orders, and the auctioneer only submits a market order when the bidding is completed. The main difference between a first price sealed bid and an oral auction is that in the latter bidders have access to the book, and consequently have reason to raise their bids. The Dutch auction is also a limit order market. In this case the auctioneer submits successively more attractive limit sell orders until a bidder submits a market buy order. Not all auctions are limit order markets however. For example the transaction price in a second price sealed bid auction is not the price of the winning limit order, but the bid of the second highest limit order, which contravenes the conventions of limit order markets. Nor is the Japanese auction a limit order market. In this case the a bidder submits successively more attractive limit order until he retires from the bidding altogether.

Retail stores set limit orders, customers make market orders.

Housing markets both buyers and sellers submit both limit and makret orders

Efficient trading mechanism: using noncredible prices as a language for communicating

5.1 Direct revelation trading game

Consider the following type of a simple trading mechanism for N players. Player $n \in \{1,\ldots,N\}$ is endowed with x_n unit of a good and some money m_n. He may use his cash to buy extra units of the good which is valued at v_n, or sell his endowment. Each person knows how to evaluate his At the beginning of the game each trader announces a valuation. Then a trading rule is picked and traders exchange according to the rule. The rule consists of a price that depends on the announcements, and partition of the players that divides them into buyers, sellers and a group of players who do not trade.

Initially we suppose that N is an odd number, that each person is endowed with cash units equal to their valuation, that is $m_n = v_n$, that the first two units of the good are valued at v_n each but further units have no value, and that traders cannot observe the cash endowments of teh other traders and do not know how they value the first two units of the good either. We consider a rule which sets the price at the median announcement m and requires traders to buy one unit if their announcement is higher than the median, and sell one unit if their announcement is lower than the median.

We now each player has a weakly dominant strategy to truthfully reveal his valuation $a_n = v_n$. Consider how the n^{th} player's payoffs would change if he deviated from this strategy by announcing $a_n = v_n^{'} \neq v_n$ instead, given any vector of announcements by the other players $a_{-n} \equiv (a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_N)$, that is irrespective of whether they are arerevealing theit true valuations or not. Denote by m' the median of the N announcments $\left(v_n^{'}, a_{-n}\right)$ when player n deviates in this fashion. If $m' = m$ the median is unaffected, and there is neither gain nor loss from announcing $v_n^{'}$ instead of v_n; the price is unchanged, and so is the direction of trade. Supposing $m' = v_n^{'}$ then player n does not trade if he announces $v_n^{'}$. Likewise he does not trade if he reveals his valuation and $v_n = m$, but gains from trading if $v_n \neq m$. The only remaining possibility is that $m' \neq m$ and $m' \neq v_n^{'}$ either; the median changes when player n announces $v_n^{'}$ instead of v_n but the valuation he announces is not the median of $\left(v_n^{'}, a_{-n}\right)$. In this case the player loses from trading by announcing $v_n^{'}$, in contrast to revealing his valuation, where the worst he can do is break even. Therefore this is a direct revelation game.

Apart from its simple structure and straightforward solution, this trading mechanism has several other desirable features. First, the mechanism is efficient. After reallocating goods and cash following the game it is impossible to improve the welfare of any player without making at least one other player worse off. Second, it is self financing, and after the common price is set, trading is fully decentralized. No subisidies or taxes are required to implement the solution, and all trades are bilateral. A third desirable feature is that the participation constraint is satisfied when trading takes place. In other words no trader would prefer to renig on the contract and not trade once the terms of trade are announced. Furthermore the median valuation player does not gain for trading at that price.

To implement this solution in a limit order market, consider the strategy. Everyone with a valuation above the population median submits a buy order a the valuation which corresponds to

$$F(b_n) = F(v_n) - F(1/2)$$

and every player with a valuation less than teh median submits a limit order to sell at

$$F(a_n) = F(v_n) + F(1/2)$$

This immediately informs everyone what the median valuation is, and trading then occurs at v_{median}, the median sample valuation, palyers with valuations less than v_{median} submitting limit orders to sell at that price, and players with valuations exceeding v_{median} submitting limit orders to buy. Figure 19.2 illustrates the history of a game for five players. Since this allocation is implemented in dominant strategies, the order in which players submit their inital bids is immaterial to the game price and quantity outcomes.

The assumptions about the trading environment can be somewhat relaxed without sacrificing much efficiency while preserving most of the simplicity and the durability of the mechanism. For example if an even number of players join the game, the two median players are not permitted to trade, and a price is set between the two median announcements, then the arguments given above can be applied to show that the mechanism is a direct revelation game and that it is self fincancing, and the participation constraint is satisfied by all trades when the terms of trade are announced . However there is an efficiency loss because the two median valuation players do not trade with each other, and at the end of the game these players have an incentive to make an exachange. Were these two palyers to break the rules of the game and trade at the end, anticipating their violation would unravel the outcome because players would not truthfully reveal their valuations in the first stage of the game. We remark that as the number of traders increases, the efficiency and concerns that players might not abide by teh rules of the mechanism diminish, because the difference between the valuations of the two median valuations declines (that is if all valuatrions are drawn from the same probability distribution), and concommitently the gain from breaking the rules also decreases. Hence the mechanism works almsot as well weh there are an even number of players as when the number of players is odd.

Similar issues arise if traders seek to buy or sell more than one unit. To preserve the truthful revelation property, we rule out the possbilty that teh annoucnement made about one item might affet the price paid for or received on other units.

5.2 Specialists

Now we turn to markets intermediated by specialists. In this case a third party intermediates trade between the buyers and sellers. This is also a very common market form in the exchange of real goods and services, for example in the real estate and car markets and retail trade.

The role of intermediaries is to promote trade by acting as middlemen between buyers and sellers. We begin our analysis by analyzing specialists, traders who enjoy a monopoly position

Trading in some industries uses middlemen to seek buyers and sellers execute trades between Intermediaries. Stocks which are only traded on the New York Stock Exchange (NYSE) functions this way. All orders must pass through a specialist, who is obliged to set bid and ask quotes that indicate at what prices buyers and sellers of the stock may trade. We model this exchange as a monopoly/monopsony. Every person seeking to change their holdings of the asset must trade with the specialist, who is free to set the offer price (at which traders can buy stock) and the bid price (at which traders may sell their stock), and change them in response to market conditions.

We first consider a private value specialist market for a security that must be traded through him.

In a market managed by a specialist, all transactions involve the specialist, who maximizes his profits by limiting trade and driving a wedge between because traders seeking increase or decrease their holdings of shares cannot trade with each other but must trade through the specialist alone.

Specialists place limit orders and investors place market orders. In this respect specialist engage in a double Dutch auctions on both sides of the market, simultaneously choosing quantity (like a monopolist who cannot commit to limit quantity)

In fact the job of a specialist is more demanding than that because he is not allowed to front run orders, he must post orders as they arise, and must cross orders with the best quotes. This means that a specialist must move quickly to keep the books clear of anything that might be crossed before a broker submits a more attractive order instead.

Competitive intermediaries

In fact the rules of the NYSE require specialists to match orders by buyers and seller without taking a portioin of the gains from trade when the orders overlap. In our setup this is achieved within a limit order market when either and/or buyers and sellers can place limit orders. Thus specialists process orders for buyers (and sellers) and in that respect perform a function that could be personally undertaken by the buyer (seller) of finding a seller (buyer). Which mechanism is more efficient Is there a role for specialization when there are several trading markets and traders have the option of spreading themselves across markets versus concentrating on a few? How about a tax on trade that is declining in volume?

we now consider a world similar to the perfect forsight trading model discussed above, at least so far as asset holders are concerned. We also introduce another class of players called dealers, auctioneers who can make limit buy and limit sell offers. Now imagfine there is competition for being a specialist. Specialists now offer a sequence of dutch auctions, reducing teh bid until a bidder takes the limit order. then speicalislists on teh other side of teh market submit increasongly attractive limit orders until soemone on the other side of the market takes it up with a market order. this consitinues because there is always an auctioneer who is willing to enter and make an exchange by first buying and then selling until there are no more gains from trade left.

the expected profits to dealers are zero. The turnover is exactly around the sameple median of teh distribution and all the gains from trade are exhausted.

Experiment 19.6

Set up an exchange in which there are shreholders and dealer/brokers. Shareholders place market orders; dealers place limit orders.

6 Dynamic Optimization and Equilibrium

The portfolio of a trader comprises the financial assets and liabilities she holds including her buy and sell orders, which while active are binding commitments to buy and sell at designated prices. At each successive instant $t \in [0,T]$ the trader may realign by submitting or withdrawing an order. Then the j^{th} asset is a stock, π_{jT} is a random variable, so it is impossible to base trading decisions at time $t < T$ on π_{jT} which unknown. At each instant t the trader maximizes the expected value of the wealth at T from her portfolio, which is:

$$E_t\left[u(x_{jT})m_T + \sum_{j=1}^{J} \pi_{jT}x_{jT}\right]$$

The J short sale constraints require:

$$\sum_{k=1}^{\infty} s_{jkt} \leq x_{jkt}$$

for each $j \in \{1,\ldots,J\}$.

A bankruptcy constraint on buy orders prevents the trader from placing orders that exceed her money holdings. It effectively constrains the seller from exchanging (selling) more money for assets than she holds. Let b_{jkt} denote the quantity of the j^{th} asset demanded at price k at time t. The single bankruptcy constraint requires:

$$\sum_{j=1}^{J}\sum_{k=1}^{\infty} p_j b_{jkt} \leq m_t$$

At each

- one of the J asset markets in which to add a new order or delete one
- the submission price p of the new order
- the quantity q, which may be negative or positive to reflect supply versus demand

to maximize (in the multiplicative case):

$$E\left\{u\left(m_T + \sum_{j=1}^{J} b_j c_{jT}x_{jT}\right)|s_{nt}\right\}$$

where trading at each instant $t \in [0,T]$ is subject to the J budget constraints preventing short sales, and the single overall budget constraint preventing borrowing.

An order strategy for the n^{th} trader is a sequence of mappings $q_n(s_{nt})$ for each $t \in [0,T]$ and $s_{nt} \in S_{nt}$. Let $q_n^o(s_{nt})$ denote the optimal order strategy, and define the value of correctly solving at time t by:

$$W(s_{nt}) \equiv \max_{q} E_t\left\{u\left(m_T + \sum_{j=1}^{J} b_j c_{jT} x_{jT}\right)\right\}$$

Then for all $\tau \in \{1, 2, \ldots, T - t\}$, the value function $W(s_{nt})$ solves the recursion:

$$W(s_{nt}) = E[W(s_{n,t+\tau})|s_{nt}]$$

The solution to limit order market games is defined in the same way we have defined the solutions to bargaining and auction games. The actions of the other players affect the trading opportunities of player $n \in \{1, \ldots, N\}$. Consequently the probability distributions the n^{th} player uses to successively take expectations over future events are partly determined by the trading strategies of the other players. In this respect our discussion ot the trader's problem above gives an incomplete picture of the trader's optimization problem, because it does not fully describe how to take the expectations over future trading opportunities. We extend the solution to the limit order market game. Let $s_n \in S_n$ denote a strategy for trader n for the whole game. It enconpasses the strategy the trader would play in a game that started at time t from information set i_{nt}, which we denoted $s_{nt}(i_n)$. We now suppose that the strategy profile $(s_1, \ldots, s_N) \in S_1 \times \ldots \times S_N$ is chosen by the players. Let $s_{-n} \equiv (s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_N)$ denote the trading strategies of all the players except the n^{th}. We write $E_s[x_{jT}|i_{nt}]$ as the expected value of x_{jT} (the value of the j^{th} asset at time T) when the traders collectively play strategy s (which this helps determine order submission flow), and the n^{th} trader has information i_{nt} at time t. We call s a solution for teh game if, for all traders $n \in \{1, \ldots, N\}$, for all points of time in the game $t \in [0, T]$, every information set i_{nt} that trader n can reach at that time, and compared to $s_n^* \in S_n$, which is any other strategy he might have selected:

$$E_s\left[u\left(m_T + \sum_{j=1}^{J} b_j c_{jT} x_{jT}\right)|s_{nt}\right] \geq E_{(s_{-n}, s_n^*)}\left[u\left(m_T + \sum_{j=1}^{J} b_j c_{jT} x_{jT}\right)|s_{nt}\right]$$

The intuitive interpretation of this inequality is that, in any solution to the game, upon reaching time t with information s_{nt}, each trader n should affirm that s_n, his strategy for the game, is at least as good as any other strategy s_n^* when the other traders are playing s_{-n}, and when his objective is to maximize the expected utility of $m_T + \sum_{j=1}^{J} b_j c_{jT} x_{jT}$ received at the terminal payoff. This solution embodies three concepts emphasized in previous chapters. First, players have rational expectations, correctly anticipating the the probability distribution characterizing future events, given the stategies of all the other players. This requirement implies that players update new information using Bayes rule. Second, a strategy is judged whether it a best response or not at each information node, not just at the beginning of the game. This requirement imposes subgame perfection on the solution to the game. Third, each trader chooses a best response to the strategies selected by the others. This requirement is the defining feature of a Nash equilibrium. The inequalites do not, however, embody the notion that weakly dominated strategies should be ruled out as solutions, although this would certainly be a reasonable requirement to add.

What can we say about individual behavrior An implication of optimality is that traders seek the least cost method for filling market orders. Similarly optimizing traders seek the best price for their assets they are selling. Thus solutions to trading games are characterized by the absence of arbitrage opportunities. In trading gmaes where traders maximize the expected value of their protflolios that are assets are linear combinations of a smaller number of factors, the absence of arbitrage opportunities have implications for the limit and market orders. Another restrictionStochastic dominance and monotonicity

6.1 The no trade theorem

Suppose each investor is privy to the returns on one of several assets but they all value them in the same way, and are risk averse. Is there scope for mutual insurance? We start our discussion with a difference in palyers that should not lead to trade, differential information about teh value of an asset. no trade theoremAn important issue in finance is teh degree to which superior information can compromise the market. It may offset the exposure of limit orders.

6.2 Arbitrage

An application of best responses and an implciation of equilibrium.

Constructing an asset portefolio embodies a particular factor mix, and two portefolios with same factor mix yield the same returns, and one might speculate trade at teh same prices.

Suppose the common value of one stock is driven by a factor π_{1t} and another is driven by the sum of the other factors. Then in a competitive equilibrium, the price of the stock which is teh sum of the other should be the sum of theri prices. More generally suppose

$$\pi_{1t} = \sum_{j=1}^{J} \alpha_j \pi_{jt}$$

then the price of the first asset should be the wrighted price of the . Indeed in evaluating Investment strategies, this is

Suppose stock returns are a linear combination of returns on some factors. That is

$$\pi_{it} = \sum_{j=1}^{J} \alpha_{ij} x_{jt}$$

where x_{jt} is the return on the j^{th} factor, and α_{ij} is the factor weight in the i^{th} stock, and for convenience we normalize the weights so that they sum to unity:

$$\sum_{j=1}^{J} \alpha_{ij} = 1$$

We define the i^{th} stock to be payoff equivalent to a portefolio of $K-1$ stocks with positive weights $(\beta_1, \ldots \beta_{i-1}, \beta_{i+1}, \ldots \beta_K)$ if the returns from holding the stock are identical to the returns from holding the portfolio at each instant in the game:

$$\alpha_{ij} = \sum_{k=1,k\neq 1}^{K} \beta_k \alpha_{kj}$$

Arbitrage is the profitable exchange of a stock for a payoff equivalent portfolio through trading. The absence of arbitrage opportunities imposes restrictions on the best quotes, the transaction prices of market orders. The market price of aquiring a unit of the i^{th} stock at time t is the best offer of the limit sell orders, denoted by s_{it}, while the market price of disposing of a unit of the k^{th} stock at time t is the best bid of the limit buy orders, denoted b_{kt}. If there are more than β_k bids at that price, and this is true for all $k \neq i$, then the market price of selling the portfolio is

$$\sum_{k=1,k\neq 1}^{K} \beta_k b_{kt}$$

In equilibrium all arbitrage opportunites are exploited, which means that for all stocks $i \in \{1, \ldots, I\}$ and all times $t \in [1, T]$

$$\sum_{k=1,k\neq 1}^{K} \beta_k b_{kt} \leq s_{it}$$

To see why the inequality will never be reversed, imagine that it is:

$$\sum_{k=1,k\neq 1}^{K} \beta_k b_{kt} > s_{it}$$

Consider the trader who has placed the best offer in By a symmetric argument, we also require

$$b_{it} \leq \sum_{k=1,k\neq 1}^{K} \beta_k s_{kt}$$

Noting that teh in limit order markets the best limit order bid exceeds the best limit order offer offer

$$b_{it} \leq \max\left\{s_{it}, \sum_{k=1,k\neq 1}^{K} \beta_k s_{kt}\right\}$$
$$s_{it} \geq \max\left\{b_{it}, \sum_{k=1,k\neq 1}^{K} \beta_k b_{kt}\right\}$$

From this inequalities it is easy to see that teh bounds on arbitrage opportunities less onorous the larger the spread. More formally, if the restrictions implied by arbitrage are met by the vector of offer and bid quotes $\{b_{kt}, s_{kt}\}_{k=1}^{I}$, then they will certainly be met if the spead is increased on any one of the stocks. Arbitrage opportunities are more likely to appear if spreads are small, because they rely on transactions, and the spread acts just like a tax on trading.

In the special case of competitive equilibrium, the differnce between the best bid and the best offer for any asset k, is vanishingly small To all intents and purposes $p_{kt} = b_{kt} = s_{kt}$. In this case it follows that

$$b_{it} = \sum_{k=1,k\neq 1}^{K} \beta_k b_{kt} = \sum_{k=1,k\neq 1}^{K} \beta_k s_{kt} = s_{it}$$

whenever these prices are defined. This result corresponds to standard results in finance on arbitrage pricing.

We have derived the arbitrage restrictions for teh most basic case, when the markets are liquid enough to buy (or sell) enough bids at the best quotes. It is notatationaly more cumbersome, but nevertheless quite intuitive to extend this concept of arbitrage to situations where undertaking the a market transaction is large enough to shift the spread. In this case we require

$$b_{it} \leq \sum_{k=1,k\neq 1}^{K} \sum_{h=1}^{\infty} q_{kh} s_{kht} 1\left\{\sum_{h=1}^{\infty} q_{kh} \leq \beta_k\right\}$$

It follows that because of discreteness the formula above might not be met, but only teh weaker analogues derived here.

Exercise *We consider several games where there are arbitrage opportunities*

1. *Consider a model where factor returns are common knowledge and the portfolio weights are known.*

2. *Consider a model where factor returns are common knowledge and portfolio weights are known by a subset of the population*

3. *Consider a model where factor returns are not common knowledge, but observed by only a subset of the poluation, and portfolio weights are known.*

4. *Show what happens as the market is less fragmented.*

6.3 Stochastic dominance

The concept of eliminating arbitrage opportunities through trade can be extended beyond financial instruments that have exactly the same factor structure. Taking advantage of arbitrage opportunities relaxes the constraints impose by the budget set. A particular kind of arbitrage opportunity occurs when one asset first order stochasitically dominates a second but is priced the same. In this case an asset with a positive price can be sold without affecting the final porbability distiribution of allocations. Second order stochastic dominance also in this case there is some Both arbitrage oppotunities and the existece of second order dominanceare inconsistent with optimization in equilibrium.

arbitrage and first order stochastic dominance

attitiude of risk averse investors that hold independently of precise form of their utility function

6.4 Monotonicity

The lack of arbitrage opportunities in equilibrium and the behavioral implications of stochastic dominance are manifested in asset pricing restrictions. The choices traders make also reveal how they value different asset allocations. We first consider a simple double auction market for two traders of the type described earlier in this chapter. The order in which the bids are opened is random with equal probability. The first order is treated as a limit order. A transaction occurs if the second order is of opposite sign to the first. The first trader places valuation v_1 on the asset, the second v_2. Consequently the net gain to the first trader is

$$1[s_2 < b_1]\left[v_1 - \left(\frac{b_1 + s_2}{2}\right)\right] + 1[s_1 < b_2]\left[\left(\frac{s_1 + b_2}{2}\right) - v_1\right]$$

Suppose the optimal bid for a a trader with valuation v_1' is b_1' and the optimal bid for a a trader with valuation v_1'' is b_1''. Denote by $F(s_2)$ the cumulative distribution function of the limit sell order by the second trader in equilibirum, and let $G(b_2)$ denote the equilibrium cumulative distribution function of her limit buy order. Integrating over (b_2, s_2) the expected net gain to the first trader from submitting (b_1, s_1) is

$$\int_0^{b_1}\left[v_1 - \left(\frac{b_1 + s_2}{2}\right)\right]dF(s_2) + \int_{s_1}^{\infty}\left[\left(\frac{s_1 + b_2}{2}\right) - v_1\right]dG(b_2)$$

Allthough the equilbrium is not uniquely determined in this model, all the equilibrium share an monotonicity property that relates valuations to limit order prices. In this context, monotonicity means that traders with higher vauations submit higher priced buy orders and also higher priced sell orders. To prove monotonicity, consider two valuations that the trader might have, v_1' or v_1''. Suppose her optimal buy order is b_1' if she has valuation v_1', and b_1'' if she has valuation v_1''. With valuation v_1' her net gain from submitting b_1' rather than b_1'' is:

$$\int_0^{b_1'}\left[v_1' - \left(\frac{b_1' + s_2}{2}\right)\right]dF(s_2) - \int_0^{b_1''}\left[v_1' - \left(\frac{b_1'' + s_2}{2}\right)\right]dF(s_2) \geq 0$$

whereas symetrically, if her valuation was v_1'' her net gain from submitting b_1'' instead of b_1' would be:

$$\int_0^{b_1''}\left[v_1'' - \left(\frac{b_1'' + s_2}{2}\right)\right]dF(s_2) - \int_0^{b_1'}\left[v_1'' - \left(\frac{b_1' + s_2}{2}\right)\right]dF(s_2) \geq 0$$

Subtracting the second inequality from the first, all the terms not involving v_1' and v_1'' cancel out, revealing a third inequality:

$$(v_1' - v_1'')\int_{b_1''}^{b_1'} dF(b_1) = (v_1' - v_1'')[F(b_1') - F(b_1'')] \geq 0$$

This inequality shows that the equilibrium limit order buy price is an increasing function of the valuation. Analogous reasoning proves that the equilibrium limit order sell price is also an an increasing function of the valuation. The logic for the second trader is symmetric. Appealing to the principle of dominance also establishes if $v_1 \leq b_1$ in equilibrium, then the probabilty of the the first trader purchasing an asset is zero, and similarly if $s_1 \leq v_1$ in equilibrium, then the probabilty of her selling the asset is zero.

In a simple double auction market there is a tight relationship between the probability of an order being executed and how attractively it is priced. Submitting a lower offer increase the probability of a sale, and submitting a higher bid increases the probability of a purchase. This intuitive result is surprisingly elusive to establish in more general trading games, where traders have the opportiunity to withdraw an inital offer and resubmit a new one. Nevertheless a monotonicty property does link valuations with execution probabilties in all the equilibrium of many trading games.

More generally orders can be ranked by their probability of execution. Let $p_{it} > p_{jt}$ for $i > j$ with $p_{0t} \equiv 1$ denoting a market order to buy. Then traders with more extreme valuations submit orders that are more likely to transact, and traders with higher valuations are more likely to submit buy orders than sell orders. If a trader with valuation v_1 submits an order at price p and

Execution probabilities and valuations directly related. Therefore terms of trade for extreme valuations are less favorable than those with less extreme valuations. Denote by P_j^b the probability that a buy order is executed for a trader $j \in \{1,2\}$ with valuation with v_j, where p_j is the expected price conditional upon execution. Suppose the gains from trade are linear in the valuation. Then teh expected gain to the j^{th} trader is $P_j^b(v_j - p_j) - \xi_j^b$, and assuming both traders are behaing optimally, it immediately follows that teh expected net gain to the first trader $P_1^b(v_1 - p_1) - \xi_1^b$ is at least as great as $P_2^b(v_1 - p_2) - \xi_2^b$, which he would have received by submitting a buy order that is optimal for a v_2 valuation trader instead:

$$P_1^b(v_1 - p_1) - \xi_1^b \geq P_2^b(v_1 - p_2) - \xi_2^b$$

Rearranging this inequality:

$$(P_1^b - P_2^b)v_1 \geq P_1^b p_1 - P_2^b p_2 + \xi_1^b - \xi_2^b$$

and, by symmetry:

$$(P_1^b - P_2^b)v_2 \leq P_1^b p_1 - P_2^b p_2 + \xi_1^b - \xi_2^b$$

Subtracting the second inequality from the first, and collecting terms we obtain

$$(P_1^b - P_2^b)(v_1 - v_2) \geq 0$$

This third inequality shows that if $P_1^b > P_2^b$ then $v_1 \geq v_2$ and vice versa. Facing exactly the same market conditions, traders with higher valuations submitting buy orders have a higher probability of excecuting their orders than traders with low valuations submitting buy orders.

The same argument can be made for sellers. Denoting by P_j^s the probability that a sell order for price p_j is executed for a trader $j \in \{1,2\}$ with valuation with v_j, anaolgous reasoning leads us to the unequality that

$$(P_1^s - P_2^s)(v_1 - v_2) \leq 0$$

In words, a trader with valuation v_1 who submits a sell order with execution probability of P_1^s has a greater chance of execution than a trader with a higher valuation also submitting a sell order if the market conditions are the same and both traders are acting optimally. another v_1.

Another possibilty is that a trader might not place an order at all. In that case the execution probablity is obviously zero. In this case we have

$$P_1^b(v_1 - v_0) \geq 0 \geq P_2^s(v_0 - v_2)$$

This leads us to obtain the expected value of the gains from trade as a function of the valuations of the traders, a convex function. Integrating integration over the state space, the expected net payoff function preserves its convexity.

We test this by forming indicator functions of whether the asset is traded or not, and then comparing this with the valuations, averaging over intervals, and comparing the resulting probabilites. When the net payoffs are computed and joined we should see apiecewise linear convex function.

Exercise *Conduct a double auction experiment between pairs of players who valuations are drawn from the* $[0,1]$ *uniform distribution. Each pair should play several rounds together before rotating to form a new pair with another bargaining partner. Pairs of to test the following hypotheses for each given trading par separately*

1. *Are bids monotone increasing in valuation?*
2. *Repeat the exercise for asks*
3. *Is the spread between asks and bids uniformly positive?*
4. *How does the probability of selling buying and not trading vary with the valuation?*

7 Summary

This chapter extended our earlier analyses of bargaining, auction and monopoly games to market settings. First we examined the effects of competition in a market for a homogeneous product as the number of suppliers increased. Initially the number of suppliers was held fixed, but then we explored what happens when entry and exit are determined within the model. In the second part of the chapter we studied how the number of sellers is determined within the equilibrium as a mapping from the underlying technology. We argued that there are essentially two case to examine, constant costs of production and declining costs of production. An important special case of the latter involves setup costs, and this is the one we focused upon. When the number of firms is endogenous there is scope for early entrants to deter potential entrants from competing in the industry. Sequential entry of traders. Then we analyzed model where there are asymmetries is chapter analyzes how the nature of strategic play changes when there is more than one player on both sides of the market. This can occur is several ways. Capacity expansion when setup costs are endogenous. Shielding the firm from information about the other firms so that response is slow to new competition, which would drive down the price of the product for several periods before a more cooperative could be reached. Third is incomplete information about the cost structure of the pioneering firm, and/or about the demand for the product itself. We also explored the effects of rivalry on the allocation of resources. Depending on the strategy played we found that in a market with heterogenious demanders, the outomce might exhaust the gains from trade or yeild a monopoly outcome, or somewhere in between.

The middle section of the paper moved away from the size of teh participants and discussed what happens when the direction of trading is endogenized. We showed that trading mechanisms exist that are fully efficient or asymptitotically efficient.

The last esctions of this chapter took an institutional approach to this problem by introducing a broad class of trading mechanisms called limit order markets. Such markets are notable for two features. The first is that every trade is the outcome of one trader proposing to trade a specified price and quantity (submitting a limit order) and another trader accepting the proposal (executing the limit order with a market order). The second feature of limit order markets is the price-time priority, which means that more higher priced limit orders ot buy and lower priced limit offers to sell respectively have prioirty over lower priced buy orders and higher priced sell offers, and if two active buy orders have teh same price, then the order submitted first receive priority. We showed that many, although by no means all trading exchanges are limit order markets. Matching games, bargaining and haggling games, most auctions and many more conventionally defined markets all fit within teh limit order market definition. Following these examples presenting a framework for designing, analyzing and conducting experiments in limit order market games to test the necessary conditions that we derived from the equilibrium conditions